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SMITHSONIAN MISCELLANEOUS COLLECTIONS
VOLUME 74, NUMBER 1

# SMITHSONIAN MATHEMATICAL FORMULAE AND TABLES OF ELLIPTIC FUNCTIONS

MATHEMATICAL FORMULAE PREPARED BY
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TABLES OF ELLIPTIC FUNCTIONS PREPARED UNDER THE DIRECTION OF
SIR GEORGE GREENHILL, BART.
BY
COL. R. L. HIPPISLEY, C.B.

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# ADVERTISEMENT

The Smithsonian Institution has maintained for many years a group of publications in the nature of handy books of information on geographical, meteorological, physical, and mathematical subjects. These include the Smithsonian Geographical Tables (third edition, reprint, 1918); the Smithsonian Meteorological Tables (fourth revised edition, 1918); the Smithsonian Physical Tables (seventh revised edition, 1921); and the Smithsonian Mathematical Tables: Hyperbolic Functions (second reprint, 1921).

The present volume comprises the most important formulae of many branches of applied mathematics, an illustrated discussion of the methods of mechanical integration, and tables of elliptic functions. The volume has been compiled by Dr. E. P. Adams, of Princeton University. Prof. F. R. Moulton, of the University of Chicago, contributed the section on numerical solution of differential equations. The tables of elliptic functions were prepared by Col. R. L. Hippisley, C. B., under the direction of Sir George Greenhill, Bart., who has contributed the introduction to these tables.

The compiler, Dr. Adams, and the Smithsonian Institution are indebted to many physicists and mathematicians, especially to Dr. H. L. Curtis and colleagues of the Bureau of Standards, for advice, criticism, and coöperation in the preparation of this volume.

CHARLES D. WALCOTT,
*Secretary of the Smithsonian Institution.*

*May, 1922.*

# PREFACE

The original object of this collection of mathematical formulae was to bring together, compactly, some of the more useful results of mathematical analysis for the benefit of those who regard mathematics as a tool, and not as an end in itself. There are many such results that are difficult to remember, for one who is not constantly using them, and to find them one is obliged to look through a number of books which may not immediately be accessible.

A collection of formulae, to meet the object of the present one, must be largely a matter of individual selection; for this reason this volume is issued in an interleaved edition, so that additions, meeting individual needs, may be made, and be readily available for reference.

It was not originally intended to include any tables of functions in this volume, but merely to give references to such tables. An exception was made, however, in favor of the tables of elliptic functions, calculated, on Sir George Greenhill's new plan, by Colonel Hippisley, which were fortunately secured for this volume, inasmuch as these tables are not otherwise available.

In order to keep the volume within reasonable bounds, no tables of indefinite and definite integrals have been included. For a brief collection, that of the late Professor B. O. Peirce can hardly be improved upon; and the elaborate collection of definite integrals by Bierens de Haan show how inadequate any brief tables of definite integrals would be. A short list of useful tables of this kind, as well as of other volumes, having an object similar to this one, is appended.

Should the plan of this collection meet with favor, it is hoped that suggestions for improving it and making it more generally useful may be received.

To Professor Moulton, for contributing the chapter on the Numerical Integration of Differential Equations, and to Sir George Greenhill, for his Introduction to the Tables of Elliptic Functions, I wish to express my gratitude. And I wish also to record my obligations to the Secretary of the Smithsonian Institution, and to Dr. C. G. Abbot, Assistant Secretary of the Institution, for the way in which they have met all my suggestions with regard to this volume.

E. P. ADAMS

PRINCETON, NEW JERSEY

## COLLECTIONS OF MATHEMATICAL FORMULAE, ETC.


B. O. Peirce: A Short Table of Integrals, Boston, 1899.

G. Petit Bois: Tables d'Integrales Indefinies, Paris, 1906.

T. J. I'A. Bromwich: Elementary Integrals, Cambridge, 1911.

D. Bierens de Haan: Nouvelles Tables d'Integrales Definies, Leiden, 1867.

E. Jahnke and F. Emde: Funktionentafeln mit Formeln und Kurven, Leipzig, 1909.

G. S. Carr: A Synopsis of Elementary Results in Pure and Applied Mathematics, London, 1880.

W. Laska: Sammlung von Formeln der reinen und angewandten Mathematik, Braunschweig, 1888–1894.

W. Ligowski: Taschenbuch der Mathematik, Berlin, 1893.

O. Th. Bürklen: Formelsammlung und Repetitorium der Mathematik, Berlin, 1922.

F. Auerbach: Taschenbuch fur Mathematiker und Physiker, 1. Jahrgang, 1909. Leipzig, 1909.

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# SYMBOLS

log — logarithm. Whenever used the Naperian logarithm is understood. To find the common logarithm to base 10:

$$\log_{10} a = 0.43429 \ldots \log a.$$
$$\log a = 2.30259 \ldots \log_{10} a.$$

$!$ — Factorial. $n!$ where $n$ is an integer denotes $1 \cdot 2 \cdot 3 \cdot 4 \ldots \ldots n$. Equivalent notation $\lfloor n$

$\neq$ — Does not equal.

$>$ — Greater than.

$<$ — Less than.

$\geqslant$ — Greater than, or equal to.

$\leqslant$ — Less than, or equal to.

$\binom{n}{k}$ — Binomial coefficient. See **1.51**.

$\rightarrow$ — Approaches.

$|\, a_{ik} \,|$ — Determinant where $a_{ik}$ is the element in the $i$th row and $k$th column,

$\frac{\partial(u_1, u_2, \ldots)}{\partial(x_1, x_2 \ldots)}$ — Functional determinant. See **1.37.**

$|\, a \,|$ — Absolute value of $a$. If $a$ is a real quantity its numerical value, without regard to sign. If $a$ is a complex quantity, $a = \alpha + i\beta$, $|\, a \,|$ = modulus of $a = +\sqrt{\alpha^2 + \beta^2}$.

$i$ — The imaginary $= +\sqrt{-1}$.

$\sum$ — Sign of summation, i.e., $\sum_{k=1}^{k=n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$.

$\prod$ — Product, i.e., $\prod_{k=1}^{k=n} (1 + kx) = (1 + x)(1 + 2x)(1 + 3x) \ldots (1 + nx)$.

# I. ALGEBRA

**1.00** Algebraic Identities.

1. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots\ldots + ab^{n-2} + b^{n-1})$.
2. $a^n \pm b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots\ldots \mp ab^{n-2} \pm b^{n-1})$.

$n$ odd: upper sign.
$n$ even: lower sign.

3. $(x+a_1)(x+a_2)\ldots\ldots(x+a_n) = x^n + P_1x^{n-1} + P_2x^{n-2} + \ldots\ldots + P_{n-1}x + P_n$.

$P_1 = a_1 + a_2 + \ldots\ldots + a_n$.
$P_k$ = sum of all the products of the $a$'s taken $k$ at a time.
$P_n = a_1a_2a_3\ldots\ldots a_n$.

4. $(a^2+b^2)(\alpha^2+\beta^2) = (a\alpha \mp b\beta)^2 + (a\beta \pm b\alpha)^2$.
5. $(a^2-b^2)(\alpha^2-\beta^2) = (a\alpha \pm b\beta)^2 - (a\beta \pm b\alpha)^2$.
6. $(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2) = (a\alpha + b\beta + c\gamma)^2 + (b\gamma - \beta c)^2 + (c\alpha - \gamma a)^2 + (a\beta - \alpha b)^2$.
7. $(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2) = (a\alpha + b\beta + c\gamma + d\delta)^2 + (a\beta - b\alpha + c\delta - d\gamma)^2 + (a\gamma - b\delta - c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta - d\alpha)^2$.
8. $(ac-bd)^2 + (ad+bc)^2 = (ac+bd)^2 + (ad-bc)^2$.
9. $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$.
10. $(a+b)(b+c)(c+a) = a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc$.
11. $(a+b)(b+c)(c+a) = bc(b+c) + ca(c+a) + ab(a+b) + 2abc$.
12. $3(a+b)(b+c)(c+a) = (a+b+c)^3 - (a^3+b^3+c^3)$.
13. $(b-a)(c-a)(c-b) = a^2(c-b) + b^2(a-c) + c^2(b-a)$.
14. $(b-a)(c-a)(c-b) = a(b^2-c^2) + b(c^2-a^2) + c(a^2-b^2)$.
15. $(b-a)(c-a)(c-b) = bc(c-b) + ca(a-c) + ab(b-a)$.
16. $(a-b)^2 + (b-c)^2 + (c-a)^2 = 2[(a-b)(a-c) + (b-a)(b-c) + (c-a)(c-b)]$.
17. $a^3(b^2-c^2) + b^3(c^2-a^2) + c^3(a^2-b^2) = (a-b)(b-c)(a-c)(ab+bc+ca)$.
18. $(a+b+c)(a^2+b^2+c^2) = bc(b+c) + ca(c+a) + ab(a+b) + a^3 + b^3 + c^3$.
19. $(a+b+c)(bc+ca+ab) = a^2(b+c) + b^2(c+a) + c^2(a+b) + 3abc$.
20. $(b+c-a)(c+a-b)(a+b-c) = a^2(b+c) + b^2(c+a) + c^2(a+b) - (a^3+b^3+c^3+2abc)$.

21. $(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 2(b^2c^2+c^2a^2+a^2b^2) - (a^4+b^4+c^4)$.

22. $(a+b+c+d)^2+(a+b-c-d)^2+(a+c-b-d)^2+(a+d-b-c)^2 = 4(a^2+b^2+c^2+d^2)$.

If $A = a\alpha+b\gamma+c\beta$

$B = a\beta+b\alpha+c\gamma$

$C = a\gamma+b\beta+c\alpha$

23. $(a+b+c)(\alpha+\beta+\gamma) = A+B+C$.

24. $[a^2+b^2+c^2-(ab+bc+ca)][\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\gamma\alpha)] = A^2+B^2+C^2-(AB+BC+CA)$.

25. $(a^3+b^3+c^3-3abc)(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma) = A^3+B^3+C^3-3ABC$.

## ALGEBRAIC EQUATIONS

**1.200** The expression

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$$

is an integral rational function, or a polynomial, of the $n$th degree in $x$.

**1.201** The equation $f(x) = 0$ has $n$ roots which may be real or complex, distinct or repeated.

**1.202** If the roots of the equation $f(x) = 0$ are $c_1, c_2, \ldots, c_n$,

$$f(x) = a_0(x-c_1)(x-c_2)\ldots\ldots(x-c_n)$$

**1.203** Symmetric functions of the roots are expressions giving certain combinations of the roots in terms of the coefficients. Among the more important are:

$$c_1+c_2+\ldots\ldots\ldots+c_n = -\frac{a_1}{a_0}$$

$$c_1c_2+c_1c_3+\ldots+c_2c_3+c_2c_4+\ldots\ldots+c_{n-1}c_n = \frac{a_2}{a_0}$$

$$c_1c_2c_3+c_1c_2c_4+\ldots+c_1c_3c_4+\ldots\ldots+c_{n-2}c_{n-1}c_n = -\frac{a_3}{a_0}$$

$$\ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

$$c_1c_2c_3\ldots\ldots c_n = (-1)^n\frac{a_n}{a_0}.$$

**1.204** Newton's Theorem. If $s_k$ denotes the sum of the $k$th powers of all the roots of $f(x) = 0$,

$$s_k = c_1^k + c_2^k + \ldots\ldots + c_n^k$$

$$1a_1 + s_1a_0 = 0$$

$$2a_2 + s_1a_1 + s_2a_0 = 0$$

$$3a_3 + s_1a_2 + s_2a_1 + s_3a_0 = 0$$

$$4a_4 + s_1a_3 + s_2a_2 + s_3a_1 + s_4a_0 = 0$$

$$\ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

or:

$$s_1 = -\frac{a_1}{a_0}$$

$$s_2 = -\frac{2a_2}{a_0} + \frac{{a_1}^2}{{a_0}^2}$$

$$s_3 = -\frac{3a_3}{a_0} + \frac{3a_1a_2}{{a_0}^2} - \frac{{a_1}^3}{{a_0}^3}$$

$$s_4 = -\frac{4a_4}{a_0} + \frac{4a_1a_3}{{a_0}^2} - \frac{4{a_1}^2a_2}{{a_0}^3} + \frac{2{a_2}^2}{{a_0}^2} + \frac{{a_1}^4}{{a_0}^4}$$

. . . . . .

. . . . . .

**1.205** If $S_k$ denotes the sum of the reciprocals of the $k$th powers of all the roots of the equation $f(x) = 0$:

$$S_k = \frac{1}{{c_1}^k} + \frac{1}{{c_2}^k} + \ldots\ldots + \frac{1}{{c_n}^k}$$

$$1a_{n-1} + S_1a_n = 0$$

$$2a_{n-2} + S_1a_{n-1} + S_2a_n = 0$$

$$3a_{n-3} + S_1a_{n-2} + S_2a_{n-1} + S_3a_n = 0$$

. . . . .

. . . . .

$$S_1 = -\frac{a_{n-1}}{a_n}$$

$$S_2 = -\frac{2a_{n-2}}{a_n} + \frac{{a^2}_{n-1}}{{a_n}^2}$$

$$S_3 = -\frac{3a_{n-3}}{a_n} + \frac{3a_{n-1}a_{n-2}}{{a_n}^2} - \frac{{a^3}_{n-1}}{{a^3}_n}$$

. . . . . .

. . . . . .

. . . . . .

**1.220** If $f(x)$ is divided by $x - h$ the result is

$$f(x) = (x - h)Q + R.$$

$Q$ is the quotient and $R$ the remainder. This operation may be readily performed as follows:

Write in line the values of $a_0$, $a_1$, . . . ., $a_n$. If any power of $x$ is missing write 0 in the corresponding place. Multiply $a_0$ by $h$ and place the product in the second line under $a_1$; add to $a_1$ and place the sum in the third line under $a_1$. Multiply this sum by $h$ and place the product in the second line under $a_2$; add to $a_2$ and place the sum in the third line under $a_2$. Continue this series of operations until the third line is full. The last term in the third line is the remainder, $R$. The first term in the third line, which is $a_0$, is the coefficient of $x^{n-1}$ in the quotient, $Q$; the second term is the coefficient of $x^{n-2}$, and so on.

**1.221** It follows from **1.220** that $f(h) = R$. This gives a convenient way of evaluating $f(x)$ for $x = h$.

**1.222** To express $f(x)$ in the form:

$$f(x) = A_0(x-h)^n + A_1(x-h)^{n-1} + \ldots + A_{n-1}(x-h) + A_n.$$

By **1.220** form $f(h) = A_n$. Repeat this process with each quotient, and the last term of each line of sums will be a succeeding value of the series of coefficients $A_n, A_{n-1}, \ldots, A_0$.

Example:

$$f(x) = 3x^5 + 2x^4 - 8x^2 + 2x - 4 \qquad h = 2$$

| | | | | | |
|---|---|---|---|---|---|
| 3 | 2 | 0 | −8 | 2 | −4 |
| | 6 | 16 | 32 | 48 | 100 |
| 3 | 8 | 16 | 24 | 50 | $96 = A_5$ |
| | 6 | 28 | 88 | 224 | |
| | 14 | 44 | 112 | $274 = A_4$ | |
| | 6 | 40 | 168 | | |
| | 20 | 84 | $280 = A_3$ | | |
| | 6 | 52 | | | |
| | 26 | $136 = A_2$ | | | |
| | 6 | | | | |
| | $32 = A_1$ | | | | |
| $3 = A_0$ | | | | | |

Thus:

$$Q = 3x^4 + 8x^3 + 16x^2 + 24x + 50$$

$$R = f(2) = 96$$

$$f(x) = 3(x-2)^5 + 32(x-2)^4 + 136(x-2)^3 + 280(x-2)^2 + 274(x-2) + 96$$

## TRANSFORMATION OF EQUATIONS

**1.230** To transform the equation $f(x) = 0$ into one whose roots all have their signs changed: Substitute $-x$ for $x$.

**1.231** To transform the equation $f(x) = 0$ into one whose roots are all multiplied by a constant, $m$: Substitute $x/m$ for $x$.

**1.232** To transform the equation $f(x) = 0$ into one whose roots are the reciprocals of the roots of the given equation: Substitute $1/x$ for $x$ and multiply by $x^n$.

**1.233** To transform the equation $f(x) = 0$ into one whose roots are all increased or diminished by a constant, $h$: Substitute $x \pm h$ for $x$ in the given equation,

the upper sign being used if the roots are to be diminished and the lower sign if they are to be increased. The resulting equation will be:

$$f(\pm h) + xf'(\pm h) + \frac{x^2}{2!}f''(\pm h) + \frac{x^3}{3!}f'''(\pm h) + \ldots = 0$$

where $f'(x)$ is the first derivative of $f(x)$, $f''(x)$, the second derivative, etc. The resulting equation may also be written:

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n = 0$$

where the coefficients may be found by the method of **1.222** if the roots are to be diminished. To increase the roots by $h$ change the sign of $h$.

## MULTIPLE ROOTS

**1.240** If $c$ is a multiple root of $f(x) = 0$, of order $m$, i.e.. repeated $m$ times, then

$$f(x) = (x - c)^m Q; \qquad R = 0$$

$c$ is also a multiple root of order $m - 1$ of the first derived equation, $f'(x) = 0$; of order $m - 2$ of the second derived equation, $f''(x) = 0$, and so on.

**1.241** The equation $f(x) = 0$ will have no multiple roots if $f(x)$ and $f'(x)$ have no common divisor. If $F(x)$ is the greatest common divisor of $f(x)$ and $f'(x)$, $f(x)/F(x) = f_1(x)$, and $f_1(x)$ will have no multiple roots.

**1.250** An equation of odd degree, $n$, has at least one real root whose sign is opposite to that of $a_n$.

**1.251** An equation of even degree, $n$, has one positive and one negative real root if $a_n$ is negative.

**1.252** The equation $f(x) = 0$ has as many real roots between $x = x_1$ and $x = x_2$ as there are changes of sign in $f(x)$ between $x_1$ and $x_2$.

**1.253** Descartes' Rule of Signs: No equation can have more positive roots than it has changes of sign from + to − and from − to +, in the terms of $f(x)$. No equation can have more negative roots than there are changes of sign in $f(-x)$.

**1.254** If $f(x) = 0$ is put in the form

$$A_0(x - h)^n + A_1(x - h)^{n-1} + \ldots + A_n = 0$$

by **1.222**, and $A_0$, $A_1$, . . . . ., $A_n$ are all positive, $h$ is an upper limit of the positive roots.

If $f(-x) = 0$ is put in a similar form, and the coefficients are all positive, $h$ is a lower limit of the negative roots.

If $f(1/x) = 0$ is put in a similar form, and the coefficients are all positive, $h$ is a lower limit of the positive roots. And with $f(-1/x) = 0$, $h$ is an upper limit of the negative roots.

**1.255** Sturm's Theorem. Form the functions:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$$
$$f_1(x) = f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \ldots + a_{n-1}$$
$$f_2(x) = -R_1 \text{ in } f(x) = Q_1f_1(x) + R_1$$
$$f_3(x) = -R_2 \text{ in } f_1(x) = Q_2f_2(x) + R_2$$
$$\ldots\ldots\ldots$$
$$\ldots\ldots\ldots$$

The number of real roots of $f(x) = 0$ between $x = x_1$ and $x = x_2$ is equal to the number of changes of sign in the series $f(x)$, $f_1(x)$, $f_2(x)$, . . . when $x_1$ is substituted for $x$ minus the number of changes of sign in the same series when $x_2$ is substituted for $x$. In forming the functions $f_1$, $f_2$, . . . . numerical factors may be introduced or suppressed in order to remove fractional coefficients.

Example:

$$f(x) = x^4 - 2x^3 - 3x^2 + 10x - 4$$
$$f_1(x) = 2x^3 - 3x^2 - 3x + 5$$
$$f_2(x) = 9x^2 - 27x + 11$$
$$f_3(x) = -8x - 3$$
$$f_4(x) = -1433$$

| | $f$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | |
|---|---|---|---|---|---|---|
| $x = -\infty$ | + | − | + | + | − | 3 changes |
| $x = 0$ | − | + | + | − | − | 2 changes |
| $x = +\infty$ | + | + | + | − | − | 1 change |

Therefore there is one positive and one negative real root.

If it can be seen that all the roots of any one of Sturm's functions are imaginary it is unnecessary to calculate any more of them after that one.

If there are any multiple roots of the equation $f(x) = 0$ the series of Sturm's functions will terminate with $f_r$, $r < n$. $f_r(x)$ is the highest common factor of $f$ and $f_1$. In this case the number of real roots of $f(x) = 0$ lying between $x = x_1$ and $x = x_2$, each multiple root counting only once, will be the difference between the number of changes of sign in the series $f, f_1, f_2, \ldots, f_r$ when $x_1$ and $x_2$ are successively substituted in them.

**1.256** Routh's rule for finding the number of roots whose real parts are positive. (Rigid Dynamics, Part II, Art. 297.)

Arrange the coefficients in two rows:

| | | | | |
|---|---|---|---|---|
| $x^n$ | $a_0$ | $a_2$ | $a_4$ | . . . . |
| $x^{n-1}$ | $a_1$ | $a_3$ | $a_5$ | . . . . |

Form a third row by cross-multiplication:

$$x^{n-2} \qquad \frac{a_1a_2 - a_0a_3}{a_1} \qquad \frac{a_1a_4 - a_0a_5}{a_1} \qquad \frac{a_1a_6 - a_0a_7}{a_1} \qquad \ldots\ldots$$

Form a fourth row by operating on these last two rows by a similar cross-multiplication. Continue this operation until there are no terms left. The number of variations of sign in the first column gives the number of roots whose real parts are positive.

If there are any equal roots some of the subsidiary functions will vanish. In place of one which vanishes write the differential coefficient of the last one which does not vanish and proceed in the same way. At the left of each row is written the power of $x$ corresponding to the first subsidiary function in that row. This power diminishes by 2 for each succeeding coefficient in the row.

Any row may be multiplied or divided by any positive quantity in order to remove fractions.

## DETERMINATION OF THE ROOTS OF AN EQUATION

**1.260** Newton's Method. If a root of the equation $f(x) = 0$ is known to lie between $x_1$ and $x_2$ its value can be found to any desired degree of approximation by Newton's method. This method can be applied to transcendental equations as well as to algebraic equations.

If $b$ is an approximate value of a root,

$$b - \frac{f(b)}{f'(b)} = c \text{ is a second approximation,}$$

$$c - \frac{f(c)}{f'(c)} = d \text{ is a third approximation.}$$

This process may be repeated indefinitely.

**1.261** Horner's Method for approximating to the real roots of $f(x) = 0$.

Let $p_1$ be the first approximation, such that $p_1 + 1 > c > p_1$, where $c$ is the root sought. The equation can always be transformed into one in which this condition holds by multiplying or dividing the roots by some power of 10 by **1.231**. Diminish the roots by $p_1$ by **1.233**. In the transformed equation

$$A_0(x - p_1)^n + A_1(x - p_1)^{n-1} + \ldots\ldots + A_{n-1}(x - p_1) + A_n = 0$$

put

$$\frac{p_2}{10} = \frac{A_n}{A_{n-1}}$$

and diminish the roots by $p_2/10$, yielding a second transformed equation

$$B_0\left(x - p_1 - \frac{p_2}{10}\right)^n + B_1\left(x - p_1 - \frac{p_2}{10}\right)^{n-1} + \ldots\ldots + B_n = 0.$$

If $B_n$ and $B_{n-1}$ are of the same sign $p_2$ was taken too large and must be diminished. Then take

$$\frac{p_3}{100} = \frac{B_n}{B_{n-1}}$$

and continue the operation. The required root will be:

$$c = p_1 + \frac{p_2}{10} + \frac{p_3}{100} + \ldots\ldots$$

**1.262** Graeffe's Method. This method determines approximate values of all the roots of a numerical equation, complex as well as real. Write the equation of the $n$th degree

$$f(x) = a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots\ldots \pm a_n = 0.$$

The product

$$f(x)\cdot f(-x) = A_0x^{2n} - A_1x^{2n-2} + A_2x^{2n-4} - \ldots\ldots \pm A_n = 0$$

contains only even powers of $x$. It is an equation of the $n$th degree in $x^2$. The coefficients are determined by

$$\begin{aligned}
A_0 &= a_0^2\\
A_1 &= a_1^2 - 2a_0a_2\\
A_2 &= a_2^2 - 2a_1a_3 + 2a_0a_4\\
A_3 &= a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_0a_6\\
A_4 &= a_4^2 - 2a_3a_5 + 2a_2a_6 - 2a_1a_7 + 2a_0a_8\\
&\ldots\ldots\\
&\ldots\ldots
\end{aligned}$$

The roots of the equation

$$A_0y^n - A_1y^{n-1} + A_2y^{n-2} - \ldots\ldots \pm A_n = 0$$

are the squares of the roots of the given equation. Continuing this process we get an equation

$$R_0u^n - R_1u^{n-1} + R_2u^{n-2} - \ldots\ldots \pm R_n = 0$$

whose roots are the $2^r$th powers of the roots of the given equation. Put $\lambda = 2^r$. Let the roots of the given equation be $c_1, c_2, \ldots\ldots, c_n$. Suppose first that

$$c_1 > c_2 > c_3 > \ldots\ldots > c_n$$

Then for large values of $\lambda$,

$$c_1^\lambda = \frac{R_1}{R_0}, \qquad c_2^\lambda = \frac{R_2}{R_1}, \qquad \ldots\ldots, \qquad c_n^\lambda = \frac{R_n}{R_{n-1}}.$$

If the roots are real they may be determined by extracting the $\lambda$th roots of these quantities. Whether they are $\pm$ is determined by taking the sign which approximately satisfies the equation $f(x) = 0$.

Suppose next that complex roots enter so that there are equalities among the absolute values of the roots. Suppose that

$$|c_1| \geqslant |c_2| \geqslant |c_3| \geqslant \ldots\ldots \geqslant |c_p|\,; \qquad |c_p| > |c_{p+1}|\,;$$
$$|c_{p+1}| \geqslant |c_{p+2}| \geqslant \ldots\ldots \geqslant |c_n|$$

Then if $\lambda$ is large enough so that $c_p\lambda^p$ is large compared to $c_{p+1}\lambda^{p+1}, c_1\lambda, c_2\lambda^2, \ldots . c_p\lambda^p$ approximately satisfy the equation:

$$R_0u^p - R_1u^{p-1} + R_2u^{p-2} - \ldots . \pm R_p = 0$$

and $c_{p+1}\lambda^{p+1}, c_{p+2}\lambda^{p+2}, \ldots , c_n\lambda^n$ approximately satisfy the equation:

$$R_pu^{n-p} - R_{p+1}u^{n-p-1} + R_{p+2}u^{n-p-2} - \ldots . \pm R_n = 0.$$

Therefore when $\lambda$ is large enough the given equation breaks down into a number of simpler equations. This stage is shown in the process of deriving the successive equations when certain of the coefficients are obtained from those of the preceding equation simply by squaring.

References: Encyklopadie der Math. Wiss. I, 1, 3a (Runge). Bairstow: Applied Aerodynamics, pp. 553–560; the solution of a numerical equation of the 8th degree is given by Graeffe's Method.

**1.270** Quadratic Equations.

$$x^2 + 2ax + b = 0.$$

The roots are:

$$x_1 = -a + \sqrt{a^2 - b}$$
$$x_2 = -a - \sqrt{a^2 - b}$$
$$x_1 + x_2 = -2a$$
$$x_1x_2 = b.$$

If $a^2 > b$ roots are real,
$a^2 < b$ roots are complex,
$a^2 = b$ roots are equal.

**1.271** Cubic equations.

$$(1)\quad x^3 + ax^2 + bx + c = 0.$$

Substitute

$$(2)\quad x = y - \frac{a}{3}$$

$$(3)\quad y^3 - 3py - 2q = 0$$

where

$$3p = \frac{a^2}{3} - b$$

$$2q = \frac{ab}{3} - \frac{2}{27}a^3 - c.$$

Roots of (3):

If $p > 0$, $q > 0$, $q^2 > p^3$

$$\cosh \phi = \frac{q}{\sqrt{p^3}}$$

$$y_1 = 2\sqrt{p}\,\cosh\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + i\sqrt{3p}\,\sinh\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - i\sqrt{3p}\,\sinh\frac{\phi}{3}.$$

If $p > 0$, $q < 0$, $q^2 > p^3$,

$$\cosh\phi = \frac{-q}{\sqrt{p^3}}$$

$$y_1 = -2\sqrt{p}\,\cosh\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + i\sqrt{3p}\,\sinh\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - i\sqrt{3p}\,\sinh\frac{\phi}{3}.$$

If $p < 0$

$$\sinh\phi = \frac{q}{\sqrt{-p^3}}$$

$$y_1 = 2\sqrt{-p}\,\sinh\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + i\sqrt{-3p}\,\cosh\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - i\sqrt{-3p}\,\cosh\frac{\phi}{3}.$$

If $p > 0$, $q^2 < p^3$,

$$\cos\phi = \frac{q}{\sqrt{p^3}}$$

$$y_1 = 2\sqrt{p}\,\cos\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + \sqrt{3p}\,\sin\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - \sqrt{3p}\,\sin\frac{\phi}{3}.$$

**1.272** Biquadratic equations.

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0.$$

Substitute

$$x = y - \frac{a_1}{a_0}$$

$$y^4 + \frac{6}{a_0^2}Hy^2 + \frac{4}{a_0^3}Gy + \frac{1}{a_0^4}F = 0$$

$$H = a_0a_2 - a_1^2$$
$$G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$$
$$F = a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4$$
$$I = a_0a_4 - 4a_1a_3 + 3a_2^2$$
$$F = a_0^2I - 3H^2$$
$$J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3$$
$$\triangle = I^3 - 27J^2 = \text{the discriminant}$$
$$G^2 + 4H^3 = a_0^2(HI - a_0J).$$

Nature of the roots of the biquadratic:

$\triangle = 0$ Equal roots are present

Two roots only equal: $I$ and $J$ are not both zero

Three roots are equal: $I = J = 0$

Two distinct pairs of equal roots: $G = 0$; $a_0^2I - 12H^2 = 0$

Four roots equal: $H = I = J = 0$.

$\triangle < 0$ Two real and two complex roots

$\triangle > 0$ Roots are either all real or all complex:

$H < 0$ and $a_0^2I - 12H^2 < 0$ Roots all real

$H > 0$ and $a_0^2I - 12H^2 > 0$ Roots all complex.

## DETERMINANTS

**1.300** A determinant of the $n$th order, with $n^2$ elements, is written:

$$\triangle = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = | a_{ij} | , \quad (i, j, = 1, 2, \ldots, n)$$

**1.301** A determinant is not changed in value by writing rows for columns and columns for rows.

**1.302** If two columns or two rows of a determinant are interchanged the resulting determinant is unchanged in value but is of the opposite sign.

**1.303** A determinant vanishes if it has two equal columns or two equal rows.

**1.304** If each element of a row or a column is multiplied by the same factor the determinant itself is multiplied by that factor.

**1.305** A determinant is not changed in value if to each element of a row or column is added the corresponding element of another row or column multiplied by a common factor.

**1.306** If each element of the $l$th row or column consists of the sum of two or more terms the determinant splits up into the sum of two or more determinants having for elements of the $l$th row or column the separate terms of the $l$th row or column of the given determinant.

**1.307** If corresponding elements of two rows or columns of a determinant have a constant ratio the determinant vanishes.

**1.308** If the ratio of the differences of corresponding elements in the $p$th and $q$th rows or columns to the differences of corresponding elements in the $r$th and $s$th rows or columns be constant the determinant vanishes.

**1.309** If $p$ rows or columns of a determinant whose elements are rational integral functions of $x$ become equal or proportional when $x = h$, the determinant is divisible by $(x - h)^{p-1}$.

## MULTIPLICATION OF DETERMINANTS

**1.320** Two determinants of equal order may be multiplied together by the scheme:

$$| a_{ij} | \times | b_{ij} | = | c_{ij} |$$

where

$$c_{ij} = a_{i1}b_{j1} + a_{i2}b_{j2} + \ldots\ldots + a_{in}b_{jn}.$$

**1.321** If the two determinants to be multiplied are of unequal order the one of lower order can be raised to one of equal order by bordering it; i.e.:

$$\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \ldots\ldots & \ldots\ldots & 0 \\ 0 & 1 & 0 & \ldots\ldots & \ldots\ldots & 0 \\ 0 & 0 & 1 & \ldots\ldots & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & a_{11} & a_{12} \ldots & a_{1n} \\ 0 & 0 & 0 & a_{21} & a_{22} \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & a_{n1} & a_{n2} \ldots & a_{nn} \end{vmatrix}$$

**1.322** The product of two determinants may be written:

$$\begin{vmatrix} a_{11} & \ldots\ldots & a_{1n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & \ldots\ldots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11} & \ldots\ldots & b_{1n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ b_{n1} & \ldots\ldots & b_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & b_{n1} & \cdots & b_{nn} \end{vmatrix}$$

## DIFFERENTIATION OF DETERMINANTS

**1.330** If the elements of a determinant, $\Delta$, are functions of a variable, $t$:

$$\frac{d\Delta}{dt} = \begin{vmatrix} a'_{11} & a_{12} & \cdots & a_{1n} \\ a'_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a'_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a'_{12} & \cdots & a_{1n} \\ a_{21} & a'_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a'_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$+ \cdots\cdots\cdots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a'_{1n} \\ a_{21} & a_{22} & \cdots & a'_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a'_{nn} \end{vmatrix}$$

where the accents denote differentiation by $t$.

## EXPANSION OF DETERMINANTS

**1.340** The complete expansion of a determinant of the $n$th order contains $n!$ terms. Each of these terms contains one element from each row and one element from each column. Any term may be obtained from the leading term:

$$a_{11}a_{22}a_{33} \cdots a_{nn}$$

by keeping the first suffixes unchanged and permuting the second suffixes among $1, 2, 3, \ldots, n$. The sign of any term is determined by the number of inversions from the second suffixes of the leading term, being positive if there is an even number of inversions and negative if there is an odd number of inversions.

**1.341** The coefficient of $a_{ij}$ when the determinant $\Delta$ is fully expanded is:

$$\frac{\partial \Delta}{\partial a_{ij}} = \Delta_{ij}.$$

$\Delta_{ij}$ is the first minor of the determinant $\Delta$ corresponding to $a_{ij}$ and is a determinant of order $n-1$. It may be obtained from $\Delta$ by crossing out the row and column which intersect in $a_{ij}$, and multiplying by $(-1)^{i+j}$.

**1.342**

$$a_{i1}\Delta_{j1} + a_{i2}\Delta_{j2} + \ldots + a_{in}\Delta_{jn} = \begin{cases} 0 & \text{if } i \neq j \\ \Delta & \text{if } i = j \end{cases}$$

$$a_{1i}\Delta_{1j} + a_{2i}\Delta_{2j} + \ldots + a_{ni}\Delta_{nj} = \begin{cases} 0 & \text{if } i \neq j \\ \Delta & \text{if } i = j \end{cases}.$$

**1.343**

$$\frac{\partial^2\Delta}{\partial a_{ij}\partial a_{kl}} = \frac{\partial\Delta_{kl}}{\partial a_{ij}} = \frac{\partial\Delta_{ij}}{\partial a_{kl}}$$

is the coefficient of $a_{ij}a_{kl}$ in the complete expansion of the determinant $\Delta$. It may be obtained from $\Delta$, except for sign, by crossing out the rows and columns which intersect in $a_{ij}$ and $a_{kl}$.

**1.344**

$$|\Delta_{ij}| \times |a_{ij}| = \Delta^n$$
$$|\Delta_{ij}| = \Delta^{n-1}.$$

The determinant $|\Delta_{ij}|$ is the reciprocal determinant to $\Delta$.

**1.345**

$$\Delta\cdot\frac{\partial^2\Delta}{\partial a_{ij}\partial a_{kl}} = \begin{vmatrix} \Delta_{ij} & \Delta_{il} \\ \Delta_{kj} & \Delta_{kl} \end{vmatrix} = \frac{\partial\Delta}{\partial a_{ij}}\frac{\partial\Delta}{\partial a_{kl}} - \frac{\partial\Delta}{\partial a_{il}}\frac{\partial\Delta}{\partial a_{kj}}.$$

**1.346**

$$\Delta^2\frac{\partial^3\Delta}{\partial a_{ij}\partial a_{kl}\partial a_{pq}} = \begin{vmatrix} \Delta_{ij} & \Delta_{il} & \Delta_{iq} \\ \Delta_{kj} & \Delta_{kl} & \Delta_{kq} \\ \Delta_{pj} & \Delta_{pl} & \Delta_{pq} \end{vmatrix}$$

**1.347**

$$\frac{\partial^2\Delta}{\partial a_{ij}\partial a_{kl}} = -\frac{\partial^2\Delta}{\partial a_{il}\partial a_{kj}}.$$

**1.348** If $\Delta = 0$,

$$\frac{\partial\Delta}{\partial a_{ij}}\frac{\partial\Delta}{\partial a_{kl}} = \frac{\partial\Delta}{\partial a_{il}}\frac{\partial\Delta}{\partial a_{kj}}.$$

**1.350** If $a_{ij} = a_{ji}$ the determinant is symmetrical. In a symmetrical determinant

$$\Delta_{ij} = \Delta_{ji}.$$

**1.351** If $a_{ij} = -a_{ji}$ the determinant is a skew determinant. In a skew determinant

$$\Delta_{ij} = (-1)^{n-1}\Delta_{ji}.$$

**1.352** If $a_{ij} = -a_{ji}$, and $a_{ii} = 0$, the determinant is a skew symmetrical determinant.

A skew symmetrical determinant of even order is a perfect square.

A skew symmetrical determinant of odd order vanishes.

**1.360** A system of linear equations:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = k_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = k_2 \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = k_n \end{aligned}$$

has a solution:

$$\Delta \cdot x_i = k_1\Delta_{1i} + k_2\Delta_{2i} + \ldots\ldots + k_n\Delta_{ni}$$

provided that

$$\Delta = |\, a_{ij} \,| \neq 0.$$

**1.361** If $\Delta = 0$, but all the first minors are not 0,

$$\Delta_{ss} \cdot x_j = x_s\Delta_{sj} + \sum_{r=1}^{n} k_r \frac{\partial^2 \Delta}{\partial a_{ss}\partial a_{rj}} \qquad (j = 1, 2, \ldots. n)$$

where $s$ may be any one of the integers $1, 2, \ldots., n$.

**1.362** If $k_1 = k_2 = \ldots\ldots = k_n = 0$, the linear equations are homogeneous, and if $\Delta = 0$,

$$\frac{x_j}{\Delta_{sj}} = \frac{x_s}{\Delta_{ss}} \qquad (j = 1, 2, \ldots n).$$

**1.363** The condition that $n$ linear homogeneous equations in $n$ variables shall be consistent is that the determinant, $\Delta$, shall vanish.

**1.364** If there are $n + 1$ linear equations in $n$ variables:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = k_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = k_2 \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = k_n \\ &c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n = k_{n+1} \end{aligned}$$

the condition that this system shall be consistent is that the determinant:

$$\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} & k_1 \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} & k_2 \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots\ldots & a_{nn} & k_n \\ c_1 & c_2 & \ldots\ldots\ldots & c_n & k_{n+1} \end{vmatrix} = 0$$

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**1.370** Functional Determinants.

If $y_1, y_2, \ldots, y_n$ are $n$ functions of $x_1, x_2, \ldots, x_n$:

$$y_k = f_k(x_1, x_2, \ldots, x_n).$$

the determinant:

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} = \left| \frac{\partial y_i}{\partial x_j} \right| = \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

is the Jacobian.

**1.371** If $y_1, y_2, \ldots, y_n$ are the partial derivatives of a function $F(x_1, x_2, \ldots, x_n)$:

$$y_i = \frac{\partial F}{\partial x_i} \quad (i = 1, 2, \ldots, n)$$

the symmetrical determinant:

$$H = \left| \frac{\partial^2 F}{\partial x_i \, \partial x_j} \right| = \frac{\partial\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2} \cdots \frac{\partial F}{\partial x_n}\right)}{\partial(x_1, x_2, \ldots, x_n)}$$

is the Hessian.

**1.372** If $y_1, y_2, \ldots, y_n$ are given as implicit functions of $x_1, x_2, \ldots, x_n$ by the $n$ equations:

$$F_1(y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n) = 0$$
$$\cdots$$
$$\cdots$$
$$F_n(y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n) = 0$$

then

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)} \div \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(y_1, y_2, \ldots, y_n)}$$

**1.373** If the $n$ functions $y_1, y_2, \ldots, y_n$ are not independent of each other the Jacobian, $J$, vanishes; and if $J = 0$ the $n$ functions $y_1, y_2, \ldots, y_n$ are not independent of each other but are connected by a relation

$$F(y_1, y_2, \ldots, y_n) = 0$$

**1.374** Covariant property. If the variables $x_1, x_2, \ldots, x_n$ are transformed by a linear substitution:

$$x_i = a_{i1}\xi_1 + a_{i2}\xi_2 + \ldots\ldots + a_{in}\xi_n \qquad (i = 1, 2, \ldots, n)$$

and the functions $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots\ldots, x_n$ become the functions $\eta_1, \eta_2, \ldots, \eta_n$ of $\xi_1, \xi_2, \ldots\ldots, \xi_n$:

$$J' = \frac{\partial(\eta_1, \eta_2, \ldots, \eta_n)}{\partial(\xi_1, \xi_2, \ldots, \xi_n)} = \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \cdot |a_{ij}|$$

or

$$J' = J \cdot |a_{ij}|$$

where $|a_{ij}|$ is the determinant or modulus of the transformation.

For the Hessian,

$$H' = H \cdot |a_{ij}|^2.$$

**1.380** To change the variables in a multiple integral:

$$I = \int \ldots\ldots \int F(y_1, y_2, \ldots, y_n) dy_1 dy_2 \ldots\ldots dy_n$$

to new variables, $x_1, x_2, \ldots, x_n$ when $y_1, y_2, \ldots, y_n$ are given functions of $x_1, x_2, \ldots, x_n$:

$$I = \int \ldots\ldots \int \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} F(x) dx_1 dx_2 \ldots\ldots dx_n$$

where $F(x)$ is the result of substituting $x_1, x_2, \ldots, x_n$ for $y_1, y_2, \ldots, y_n$ in $F(y_1, y_2, \ldots, y_n)$.

## PERMUTATIONS AND COMBINATIONS

**1.400** Given $n$ different elements. Represent each by a number, $1, 2, 3, \ldots, n$. The number of permutations of the $n$ different elements is,

$$_nP_n = n!$$

e.g., $n = 3$:

$$(123), (132), (213), (231), (312), (321) = 6 = 3!$$

**1.401** Given $n$ different elements. The number of permutations in groups of $r$ $(r<n)$, or the number of $r$-permutations, is,

$$_nP_r = \frac{n!}{(n-r)!}$$

e.g., $n = 4$, $r = 3$:

(123)(132)(124)(142)(134)(143)(234)(243)(231)(213)(214)(241)(341)(314)
(312)(321)(324)(342)(412)(421)(431)(413)(423)(432) = 24

**1.402** Given $n$ different elements. The number of ways they can be divided into $m$ specified groups, with $x_1, x_2, \ldots\ldots, x_m$ in each group respectively, $(x_1 + x_2 + \ldots\ldots + x_m) = n$ is

$$\frac{n!}{x_1!x_2!\ldots\ldots x_m!}$$

e.g., $n = 6$, $m = 3$, $x_1 = 2$, $x_2 = 3$, $x_3 = 1$:

| | | |
|---|---|---|
| (12) (345) (6) | (13) (245) (6) | $\times 6 = 60$ |
| (23) (145) (6) | (24) (135) (6) | |
| (34) (125) (6) | (35) (124) (6) | |
| (45) (123) (6) | (25) (234) (6) | |
| (14) (235) (6) | (15) (234) (6) | |

**1.403** Given $n$ elements of which $x_1$ are of one kind, $x_2$ of a second kind, $\ldots\ldots\ldots$, $x_m$ of an $m$th kind. The number of permutations is

$$\frac{n!}{x_1!x_2!\ldots\ldots\ldots x_m!}$$

$$x_1 + x_2 + \ldots\ldots\ldots + x_m = n.$$

**1.404** Given $n$ different elements. The number of ways they can be permuted among $m$ specified groups, when blank groups are allowed, is

$$\frac{(m+n-1)!}{(m-1)!}$$

e.g., $n = 3$, $m = 2$:

(123,0)(132,0)(213,0)(231,0)(312,0)(321,0)(12,3)(21,3)(13,2)(31,2)(23,1)
(32,1)(1,23)(1,32)(2,31)(2,13)(3,12)(3,21)(0,123)(0,213)(0,132)(0,231)
(0,312)(0,321) = 24

**1.405** Given $n$ different elements. The number of ways they can be permuted among $m$ specified groups, when blank groups are not allowed, so that each group contains at least one element, is

$$\frac{n!(n-1)!}{(n-m)!(m-1)!}$$

e.g., $n = 3$, $m = 2$:

(12,3)(21,3)(13,2)(31,2)(23,1)(32,1)(1,23)(1,32)(2,31)(2,13)(3,12)(3,21) = 12

**1.406** Given $n$ different elements. The number of ways they can be combined into $m$ specified groups when blank groups are allowed is

$$m^n$$

e.g., $n = 3$, $m = 2$:

(123,0)(12,3)(13,2)(23,1)(1,23)(2,31)(3,12)(0,123) = 8

**1.407** Given $n$ similar elements. The number of ways they can be combined into $m$ different groups when blank groups are allowed is

$$\frac{(n+m-1)!}{(m-1)!n!}$$

e.g., $n = 6$, $m = 3$:

Group 1 6 5 5 4 4 4 3 3 3 3 2 2 2 2 2 1 1 1 1 1 1 0 0 0 0 0 0 0
Group 2 0 1 0 2 0 1 3 0 2 1 4 0 3 1 2 5 0 4 1 3 2 6 0 5 1 4 2 3 } = 28
Group 3 0 0 1 0 2 1 0 3 1 2 0 4 1 3 2 0 5 1 4 2 3 0 6 1 5 2 4 3

**1.408** Given $n$ similar elements. The number of ways they can be combined into $m$ different groups when blank groups are not allowed, so that each group shall contain at least one element, is

$$\frac{(n-1)!}{(m-1)!(n-m)!}.$$

## BINOMIAL COEFFICIENTS

**1.51**

1. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} = {}_nC_k = \frac{n(n-1)(n-2)\ \ldots\ (n-k+1)}{k!}.$

2. $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$

3. $\binom{n}{0} = 1,\ \binom{n}{1} = n,\ \binom{n}{n} = 1.$

4. $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}.$

5. $\binom{n}{k} = 0$ if $n < k$.

6. $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}.$

7. $1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^k\binom{n}{k} = (-1)^k\binom{n-1}{k}.$

8. $\binom{n}{k} + \binom{n}{k-1}\binom{r}{1} + \binom{n}{k-2}\binom{r}{2} + \ldots + \binom{r}{k} = \binom{n+r}{k}.$

9. $1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n.$

10. $1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0.$

11. $1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$

**1.52** Table of Binomial Coefficients.

$$\binom{n}{1} = n.$$

| $\binom{n}{1}$ | $\binom{n}{2}$ | $\binom{n}{3}$ | $\binom{n}{4}$ | $\binom{n}{5}$ | $\binom{n}{6}$ | $\binom{n}{7}$ | $\binom{n}{8}$ | $\binom{n}{9}$ | $\binom{n}{10}$ | $\binom{n}{11}$ | $\binom{n}{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | |
| 2 | 1 | | | | | | | | | | |
| 3 | 3 | 1 | | | | | | | | | |
| 4 | 6 | 4 | 1 | | | | | | | | |
| 5 | 10 | 10 | 5 | 1 | | | | | | | |
| 6 | 15 | 20 | 15 | 6 | 1 | | | | | | |
| 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | | |
| 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | | |
| 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | | |
| 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | | |
| 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 | |
| 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 | 1 |

**1.521** Glaisher, Mess. of Math. 47, p. 97, 1918, has given a complete table of binomial coefficients, from $n = 2$ to $n = 50$, and $k = 0$ to $k = n$.

**1.61** Resolution into Partial Fractions.

If $F(x)$ and $f(x)$ are two polynomials in $x$ and $f(x)$ is of higher degree than $F(x)$,

$$\frac{F(x)}{f(x)} = \sum \frac{F(a)}{\phi(a)} \frac{1}{x-a} + \sum \frac{1}{(p-1)!} \frac{d^{p-1}}{dc^{p-1}} \left[ \frac{F(c)}{\phi(c)} \frac{1}{x-c} \right]$$

where

$$\phi(a) = \left[ \frac{f(x)}{x-a} \right]_{x=a},$$

$$\phi(c) = \left[ \frac{f(x)}{(x-c)^p} \right]_{x=c}.$$

The first summation is to be extended for all the simple roots, $a$, of $f(x)$ and the second summation for all the multiple roots, $c$, of order $p$, of $f(x)$.

## FINITE DIFFERENCES AND SUMS.

**1.811** Definitions.

1. $\Delta f(x) = f(x+h) - f(x)$.
2. $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$.
   $= f(x+2h) - 2f(x+h) + f(x)$.

3. $\Delta^3 f(x) = \Delta^2 f(x+h) - \Delta^2 f(x)$.
$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$.

. . . . . .

. . . . . .

4. $\Delta^n f(x) = f(x+nh) - \frac{n}{1} f(x+\overline{n-1}h) + \frac{n(n-1)}{2!} f(x+\overline{n-2}h) - \ldots .$
$+ (-1)^n f(x)$.

**1.812**

1. $\Delta[cf(x)] = c\Delta f(x)$ ($c$ a constant).
2. $\Delta[f_1(x) + f_2(x) + \ldots .] = \Delta f_1(x) + \Delta f_2(x) + \ldots .$
3. $\Delta[f_1(x) \cdot f_2(x)] = f_1(x) \cdot \Delta f_2(x) + f_2(x+h) \cdot \Delta f_1(x)$
$= f_1(x) \cdot \Delta f_2(x) + f_2(x) \cdot \Delta f_1(x) + \Delta f_1(x) \cdot \Delta f_2(x)$.
4. $\Delta \frac{f_1(x)}{f_2(x)} = \frac{f_2(x) \cdot \Delta f_1(x) - f_1(x) \cdot \Delta f_2(x)}{f_2(x) \cdot f_2(x+h)}$.

**1.813** The $n$th difference of a polynomial of the $n$th degree is constant. If
$f(x) = a_0 x_n + a_1 x^{n-1} + \ldots . + a_{n-1} x + a_n$
$\Delta^n f(x) = n! a_0 h^n$.

**1.82**

1. $\frac{\Delta^m \{(x-b)(x-b-h)(x-b-2h) \ldots . (x-b-\overline{n-1}h)\}}{n(n-1)(n-2) \ldots . (n-m+1)h^m}$
$= (x-b)(x-b-h)(x-b-2h) \ldots . (x-b-\overline{n-m-1}h)$.

2. $\Delta^m \frac{1}{(x+b)(x+b+h)(x+b+2h) \ldots . (x+b+\overline{n-1}h)}$
$= (-1)^m \frac{n(n+1)(n+2) \ldots . (n+m-1)h^m}{(x+b)(x+b+h)(x+b+2h) \ldots . (x+b+\overline{n+m-1}h)}$.

3. $\Delta^m a^x = (a^h - 1)^m a^x$

4. $\Delta \log f(x) = \log \left(1 + \frac{\Delta f(x)}{f(x)}\right)$.

5. $\Delta^m \sin(cx+d) = \left(2 \sin \frac{ch}{2}\right)^m \sin\left(cx + d + m \frac{ch+\pi}{2}\right)$.

6. $\Delta^m \cos(cx+d) = \left(2 \sin \frac{ch}{2}\right)^m \cos\left(cx + d + m \frac{ch+\pi}{2}\right)$.

**1.83** Newton's Interpolation Formula.

$$f(x) = f(a) + \frac{x-a}{h}\Delta f(a) + \frac{(x-a)\,(x-a-h)}{2!\,h^2}\Delta^2 f(a) +$$

$$+ \frac{(x-a)\,(x-a-h)\,(x-a-2h)}{3!\,h^3}\Delta^3 f(a) + \ldots\ldots$$

$$+ \frac{(x-a)\,(x-a-h)\ldots\ldots(x-a-\overline{n-1}h)}{n!\,h^n}\Delta^n f(a)$$

$$+ \frac{(x-a)\,(x-a-h)\ldots\ldots(x-a-nh)}{n+1!} f^{(n+1)}(\xi)$$

where $\xi$ has a value intermediate between the greatest and least of $a$, $(a + nh)$, and $x$.

**1.831**

$$f(a+nh) = f(a) + \frac{n}{1!}\Delta f(a) + \frac{n(n-1)}{2!}\Delta^2 f(a) + \frac{n(n-1)\,(n-2)}{3!}\Delta^3 f(a)$$

$$+ \ldots\ldots + n\Delta^{n-1} f(a) + \Delta^n f(a).$$

**1.832** Symbolically

1. $\Delta = e^{h\frac{\partial}{\partial x}} - 1$

2. $f(a+nh) = (1+\Delta)^n f(a)$

**1.833** If $u_0 = f(a)$, $u_1 = f(a+h)$, $u_2 = f(a+2h)$, . . . ., $u_x = f(a+xh)$,

$$u_x = (1+\Delta)^x u_0 = e^{hx\frac{\partial}{\partial x}} u_0.$$

**1.840** The operator inverse to the difference, $\Delta$, is the sum, $\Sigma$.

$$\Sigma = \Delta^{-1} = \frac{1}{e^{\lambda\frac{\partial}{\partial x}} - 1}.$$

**1.841** If $\Delta F(x) = f(x)$,

$$\Sigma f(x) = F(x) + C,$$

where $C$ is an arbitrary constant.

**1.842**

1. $\Sigma c f(x) = c\Sigma f(x)$.
2. $\Sigma[f_1(x) + f_2(x) + \ldots] = \Sigma f_1(x) + \Sigma f_2(x) + \ldots$
3. $\Sigma[f_1(x)\cdot\Delta f_2(x)] = f_1(x)\cdot f_2(x) - \Sigma[f_2(x+h)\cdot\Delta f_1(x)]$.

**1.843** Indefinite Sums.

1. $\Sigma[(x-b)(x-b-h)(x-b-2h) \ldots (x-b-\overline{n-1}h)]$

$$= \frac{1}{(n+1)h}(x-b)(x-b-h) \ldots\ldots (x-b-nh) + C.$$

2. $$\sum \frac{1}{(x+b)(x+b+h) \ldots\ldots (x+b+\overline{n-1}h)}$$

$$= -\frac{1}{(n-1)h} \frac{1}{(x+b)(x+b+h) \ldots\ldots (x+b+\overline{n-2}h)} + C.$$

3. $$\sum a^x = \frac{a^x}{a^h - 1} + C.$$

4. $$\sum \cos(cx+d) = \frac{\sin\left(cx - \frac{ch}{2} + d\right)}{2 \sin \frac{ch}{2}} + C.$$

5. $$\sum \sin(cx+d) = -\frac{\cos\left(cx - \frac{ch}{2} + d\right)}{2 \sin \frac{ch}{2}} + C.$$

**1.844** If $f(x)$ is a polynomial of degree $n$,

$$\sum a^x f(x) = \frac{a^x}{a^h - 1}\left\{f(x) - \frac{a^h}{a^h-1}\Delta f(x) + \left(\frac{a^h}{a^h-1}\right)^2 \Delta^2 f(x) - \ldots\ldots\right.$$
$$\left. + \left(\frac{-a^h}{a^h-1}\right)^n \Delta^n f(x)\right\} + C.$$

**1.845** If $f(x)$ is a polynomial of degree $n$,

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1}x + a_n,$$

and

$$\Sigma f(x) = F(x) + C,$$
$$F(x) = c_0 x^{n+1} + c_1 x^n + c_2 x^{n-1} + \ldots\ldots + c_n x + c_{n+1},$$

where

$$(n+1)hc_0 = a_0$$
$$\frac{(n+1)n}{2!}h^2 c_0 + nhc_1 = a_1$$
$$\frac{(n+1)n(n-1)}{3!}h^3 c_0 + \frac{n(n-1)}{2!}h^2 c_1 + (n-1)hc_2 = a_2$$
. . . . . .
. . . . . .

The coefficient $c_{n+1}$ may be taken arbitrarily.

**1.850** Definite Sums. From the indefinite sum,

$$\Sigma f(x) = F(x) + C,$$

a definite sum is obtained by subtraction,

$$\sum_{a+mh}^{a+nh} f(x) = F(a + nh) - F(a + mh).$$

**1.851**

$$\sum_{a}^{a+nh} f(x) = f(a) + f(a + h) + f(a + 2h) + \ldots + f(a + \overline{n - 1}h)$$
$$= F(a + nh) - F(a).$$

By means of this formula many finite sums may be evaluated.

**1.852**

$$\sum_{a}^{a+nh} (x - b)(x - b - h)(x - b - 2h) \ldots (x - b - \overline{k - 1}h)$$
$$= \frac{(a - b + nh)(a - b + \overline{n - 1}h) \ldots (a - b + \overline{n - k}h)}{(k + 1)h}$$
$$- \frac{(a - b)(a - b - h) \ldots (a - b - kh)}{(k + 1)h}.$$

**1.853**

$$\sum_{a}^{a+nh} (x - a)(x - a - h) \ldots (x - a - \overline{k - 1}h)$$
$$= \frac{n(n - 1)(n - 2) \ldots (n - k)}{(k + 1)} h^k.$$

**1.854** If $f(x)$ is a polynomial of degree $m$ it can be expressed:

$$f(x) = A_0 + A_1(x - a) + A_2(x - a)(x - a - h) + \ldots$$
$$+ A_m(x - a)(x - a - h) \ldots (x - a - \overline{m - 1}h),$$

$$\sum_{a}^{a+nh} f(x) = A_0 n + A_1 \frac{n(n - 1)}{2} h + A_2 \frac{n(n - 1)(n - 2)}{3} h^2$$
$$+ A_m \frac{n(n - 1) \ldots (n - m)}{(m + 1)} h^m.$$

**1.855** If $f(x)$ is a polynomial of degree $(m - 1)$ or lower, it can be expressed:

$$f(x) = A_0 + A_1(x + mh) + A_2(x + mh)(x + \overline{m - 1}h)$$
$$+ \ldots + A_{m-1}(x + mh) \ldots (x + 2h)$$

and,

$$\sum_{a}^{a+nh} \frac{f(x)}{x(x + h)(x + 2h) \ldots (x + mh)} = \frac{A_0}{mh} \left\{ \frac{1}{a(a + h) \ldots (a + \overline{m - 1}h)} \right.$$

$$- \frac{1}{(a+nh) \ldots \overline{(a+n+m-1h)}} \Big\}$$

$$+ \frac{A_1}{(m-1)h} \left\{ \frac{1}{a(a+h)\ldots\overline{(a+m-2h)}} - \frac{1}{(a+nh)\ldots\overline{(a+n+m-2h)}} \right\}$$

$$+ \ldots\ldots + \frac{A_{m-1}}{h} \left\{ \frac{1}{a} - \frac{1}{a+nh} \right\}.$$

**1.856** If $f(x)$ is a polynomial of degree $m$ it can be expressed:

$$f(x) = A_0 + A_1(x+mh) + A_2(x+mh)(x+\overline{m-1}h) + \ldots\ldots + A_m(x+mh)\ldots(x+h)$$

and,

$$\sum_a^{a+nh} \frac{f(x)}{x(x+h)\ldots(x+mh)} = \frac{A_0}{mh} \left\{ \frac{1}{a(a+h)\ldots\overline{(a+m-1h)}} - \frac{1}{(a+nh)\ldots\overline{(a+m+n-1h)}} \right\}$$

$$+ \ldots\ldots + \frac{A_{m-1}}{h} \left\{ \frac{1}{a} - \frac{1}{a+nh} \right\} + A_m \sum_a^{a+nh} \frac{1}{x}$$

where,

$$\sum_a^{a+nh} \frac{1}{x} = \frac{1}{a} + \frac{1}{a+h} + \frac{1}{a+2h} + \ldots\ldots + \frac{1}{a+\overline{n-1}h}.$$

**1.86** Euler's Summation Formula.

$$\sum_a^b f(x) = \frac{1}{h}\int_a^b f(z)dz + A_1\Big\{f(b) - f(a)\Big\} + A_2 h\Big\{f'(b) - f'(a)\Big\},$$

$$+ \ldots\ldots + A_{m-1}h^{m-2}\{f^{(m-2)}(b) - f^{(m-2)}(a)\},$$

$$- \int_0^h \phi_m(z) \sum_{x=a}^{x=b} \frac{d^m f(x+h-z)}{h dx^m} \cdot dz$$

$$\phi_m(z) = \frac{z^m}{m!} + A_1 \frac{hz^{m-1}}{(m-1)!} + A_2 \frac{h^2 z^{m-2}}{(m-2)!} + \ldots\ldots + A_{m-1}h^{m-1}z.$$

$m!\phi_m(z)$, with $h = 1$, is the Bernoullian polynomial.

$A_1 = -\frac{1}{2}$, $A_{2k+1} = 0$; the coefficients $A_{2k}$ are connected with Bernoulli's numbers (**6.902**), $B_k$, by the relation,

$$A_{2k} = (-1)^{k+1}\frac{B_k}{(2k)!}$$

$$A_1 = -\frac{1}{2}, \qquad A_2 = \frac{1}{12}, \qquad A_4 = -\frac{1}{720}, \qquad A_6 = \frac{1}{30240} \cdot \ldots\ldots$$

**1.861**

$$\sum_a^b f(x) = \frac{1}{h}\int_a^b f(z)dz - \frac{1}{2}\Big\{f(b) - f(a)\Big\} + \frac{h}{12}\Big\{f'(b) - f'(a)\Big\}$$

$$- \frac{h^3}{720}\Big\{f'''(b) - f'''(a)\Big\} + \frac{h^5}{30240}\Big\{f^{v}(b) - f^{v}(a)\Big\} - \cdots\cdots$$

**1.862**

$$\sum u_x = C + \int u_x dx - \frac{1}{2}u_x + \frac{1}{12}\frac{du_x}{dx} - \frac{1}{720}\frac{d^3u_x}{dx^3} + \frac{1}{30240}\frac{d^5u_x}{dx^5} - \cdots\cdots$$

## SPECIAL FINITE SERIES

**1.871** Arithmetical progressions. If $s$ is the sum, $a$ the first term, $\delta$ the common difference, $l$ the last term, and $n$ the number of terms,

$$s = a + (a + \delta) + (a + 2\delta) + \cdots\cdots[a + (n - 1)\delta]$$

$$l = a + (n - 1)\,\delta$$

$$s = \frac{n}{2}[2a + (n - 1)\delta]$$

$$= \frac{n}{2}(a + l).$$

**1.872** Geometrical progressions.

$$s = a + ap + ap^2 + \cdots\cdots + ap^{n-1}$$

$$s = a\frac{p^n - 1}{p - 1}$$

If $p<1$, $n = \infty$, $s = \dfrac{a}{1 - p}$.

**1.873** Harmonical progressions. $a, b, c, d, \ldots$ form an harmonical progression if the reciprocals, $1/a$, $1/b$, $1/c$, $1/d$, $\ldots$ form an arithmetical progression.

**1.874.**

1. $\displaystyle\sum_{x=1}^{x=n} x = \frac{n(n+1)}{2}$

2. $\displaystyle\sum_{x=1}^{x=n} x^2 = \frac{n(n+1)\,(2n+1)}{6}$

3. $\displaystyle\sum_{x=1}^{x=n} x^3 = \left[\frac{n(n+1)}{2}\right]^2$

4. $\displaystyle\sum_{x=1}^{x=n} x^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

**1.875** In general,

$$\sum_{x=1}^{x=n} x^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \frac{1}{2}\binom{k}{1} B_1 n^{k-1} - \frac{1}{4}\binom{k}{3} B_2 n^{k-3} + \frac{1}{6}\binom{k}{5} B_3 n^{k-5} - \ldots$$

$B_1$, $B_2$, $B_3$, . . . are Bernoulli's numbers (**6.902**), $\binom{k}{h}$ are the binomial coefficients (**1.51**); the series ends with the term in $n$ if $k$ is even, and with the term in $n^2$ if $k$ is odd.

**1.876**

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \gamma + \log n + \frac{1}{2n} - \frac{a_2}{n(n+1)} - \frac{a_3}{n(n+1)(n+2)} - \ldots$$

$\gamma$ = Euler's constant = 0.5772156649 . . .

$$a_2 = \frac{1}{12}$$

$$a_3 = \frac{1}{12}$$

$$a_4 = \frac{19}{80} \qquad a_k = \frac{1}{k}\int_0^1 x(1-x)(2-x)\ldots(k-1-x)dx$$

$$a_5 = \frac{9}{20}$$

**1.877**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{b_1}{n+1} - \frac{b_2}{(n+1)(n+2)} - \frac{b_3}{(n+1)(n+2)(n+3)} - \ldots$$

$$b_k = \frac{(k-1)!}{k}$$

**1.878**

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} = C - \frac{c_2}{(n+1)(n+2)} - \frac{c_3}{(n+1)(n+2)(n+3)} - \ldots$$

$$C = \sum_{k=1}^{\infty} \frac{1}{k^3} = 1.2020569032$$

$$c_k = \frac{(k-1)!}{k}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k-1}\right).$$

**1.879** Stirling's Formula.

$$\log(n!) = \log\sqrt{2\pi} + \left(n+\frac{1}{2}\right)\log n - n + \frac{A_2}{n} + \ldots + A_{2k-2}\frac{(2k-4)!}{n^{2k-3}} + \theta A_{2k}\frac{(2k-2)!}{n^{2k-1}}$$

$0<\theta<1$. The coefficients $A_k$ are given in **1.86.**

**1.88**

1. $1 + 1! + 2\cdot 2! + 3\cdot 3! + \ldots + n\cdot n! = (n+1)!$

2. $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \ldots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3).$

3. $1\cdot 2\cdot 3\ldots r + 2\cdot 3\cdot 4\ldots(r+1) + \ldots\ldots\ldots + n(n+1)(n+2)\ldots(n+r-1) = \frac{n(n+1)(n+2)\ldots(n+r)}{r+1}.$

4. $1\cdot p + 2(p+1) + 3(p+2) + \ldots\ldots\ldots + n(p+n-1) = \frac{1}{6}n(n+1)(3p+2n-2).$

5. $p\cdot q + (p-1)(q-1) + (p-2)(q-2) + \ldots\ldots\ldots (p-n)(q-n) = \frac{1}{6}n[6pq - (n-1)(3p+3q-2n+1)].$

6. $1 + \frac{b}{a} + \frac{b(b+1)}{a(a+1)} + \ldots + \frac{b(b+1)\ldots(b+n-1)}{a(a+1)\ldots(a+n-1)} = \frac{b(b+1)\ldots(b+n)}{(b+1-a)a(a+1)\ldots(a+n-1)} - \frac{a-1}{b+1-a}.$

# II. GEOMETRY

**2.00** Transformation of coördinates in a plane.

**2.001** Change of origin. Let $x$, $y$ be a system of *rectangular* or *oblique* coördinates with origin at $O$. Referred to $x$, $y$ the coördinates of the new origin $O'$ are $a$, $b$. Then referred to a parallel system of coördinates with origin at $O'$ the coördinates are $x'$, $y'$.

$$x = x' + a$$
$$y = y' + b.$$

**2.002** Origin unchanged. Directions of axes changed. Oblique coördinates. Let $\omega$ be the angle between the $x - y$ axes measured counter-clockwise from the $x$- to the $y$-axis. Let the $x'$-axis make an angle $\alpha$ with the $x$-axis and the $y'$-axis an angle $\beta$ with the $x$-axis. All angles are measured counter-clockwise from the $x$-axis. Then

$$x \sin \omega = x' \sin (\omega - \alpha) + y' \sin (\omega - \beta)$$
$$y \sin \omega = x' \sin \alpha + y' \sin \beta$$
$$\omega' = \beta - \alpha.$$

**2.003** Rectangular axes. Let both new and old axes be rectangular, the new axes being turned through an angle $\theta$ with respect to the old axes. Then $\omega = \frac{\pi}{2}$, $\alpha = \theta$, $\beta = \frac{\pi}{2} + \theta$.

$$x = x' \cos \theta - y' \sin \theta$$
$$y = x' \sin \theta + y' \cos \theta.$$

**2.010** Polar coördinates. Let the $y$-axis make an angle $\omega$ with the $x$-axis and let the $x$-axis be the initial line for a system of polar coördinates $r$, $\theta$. All angles are measured in a counter-clockwise direction from the $x$-axis.

$$x = \frac{r \sin (\omega - \theta)}{\sin \omega}$$

$$y = r \frac{\sin \theta}{\sin \omega}.$$

**2.011** If the $x$, $y$ axes are rectangular, $\omega = \frac{\pi}{2}$,

$$x = r \cos \theta$$
$$y = r \sin \theta.$$

**2.020** Transformation of coördinates in three dimensions.

**2.021** Change of origin. Let $x, y, z$ be a system of *rectangular* or *oblique* coördinates with origin at $O$. Referred to $x, y, z$ the coördinates of the new origin $O'$ are $a, b, c$. Then referred to a parallel system of coördinates with origin at $O'$ the coördinates are $x', y', z'$.

$$x = x' + a$$
$$y = y' + b$$
$$z = z' + c$$

**2.022** Transformation from one to another rectangular system. Origin unchanged. The two systems are $x, y, z$ and $x'\ y'\ z'$.

Referred to $x, y, z$ the direction cosines of $x'$ are $l_1, m_1, n_1$
Referred to $x, y, z$ the direction cosines of $y'$ are $l_2, m_2, n_2$
Referred to $x, y, z$ the direction cosines of $z'$ are $l_3, m_3, n_3$

The two systems are connected by the scheme:

| | $x'$ | $y'$ | $z'$ |
|---|---|---|---|
| $x$ | $l_1$ | $l_2$ | $l_3$ |
| $y$ | $m_1$ | $m_2$ | $m_3$ |
| $z$ | $n_1$ | $n_2$ | $n_3$ |

$$x = l_1x' + l_2y' + l_3z' \qquad x' = l_1x + m_1y + n_1z$$
$$y = m_1x' + m_2y' + m_3z' \qquad y' = l_2x + m_2y + n_2z$$
$$z = n_1x' + n_2y' + n_3z' \qquad z' = l_3x + m_3y + n_3z$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad l_1^2 + l_2^2 + l_3^2 = 1$$
$$l_2^2 + m_2^2 + n_2^2 = 1 \qquad m_1^2 + m_2^2 + m_3^2 = 1$$
$$l_3^2 + m_3^2 + n_3^2 = 1 \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$l_1m_1 + l_2m_2 + l_3m_3 = 0 \qquad l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$m_1n_1 + m_2n_2 + m_3n_3 = 0 \qquad l_2l_3 + m_2m_3 + n_2n_3 = 0$$
$$n_1l_1 + n_2l_2 + n_3l_3 = 0 \qquad l_3l_1 + m_3m_1 + n_3n_1 = 0$$

**2.023** If the transformation from one to another rectangular system is a rotation through an angle $\theta$ about an axis which makes angles $\alpha, \beta, \gamma$ with $x, y, z$ respectively,

$$2\cos\theta = l_1 + m_2 + n_3 - 1$$

$$\frac{\cos^2 \alpha}{m_2 + n_3 - l_1 - 1} = \frac{\cos^2 \beta}{n_3 + l_1 - m_2 - 1} = \frac{\cos^2 \gamma}{l_1 + m_2 - n_3 - 1}$$

**2.024** Transformation from a rectangular to an oblique system. $x, y, z$ rectangular system: $x', y', z'$ oblique system.

$$\begin{aligned} \cos \widehat{xx'} &= l_1 & \cos \widehat{xy'} &= l_2 & \cos \widehat{xz'} &= l_3 \\ \cos \widehat{yx'} &= m_1 & \cos \widehat{yy'} &= m_2 & \cos \widehat{yz'} &= m_3 \\ \cos \widehat{zx'} &= n_1 & \cos \widehat{zy'} &= n_2 & \cos \widehat{zz'} &= n_3 \end{aligned}$$

$$\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' \\ y &= m_1 x' + m_2 y' + m_3 z' \\ z &= n_1 x' + n_2 y' + n_3 z' \end{aligned}$$

$$\begin{aligned} \cos \widehat{y'z'} &= l_2 l_3 + m_2 m_3 + n_2 n_3 \\ \cos \widehat{z'x'} &= l_3 l_1 + m_3 m_1 + n_3 n_1 \\ \cos \widehat{x'y'} &= l_1 l_2 + m_1 m_2 + n_1 n_2 \end{aligned}$$

$$\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1 \\ l_2^2 + m_2^2 + n_2^2 &= 1 \\ l_3^2 + m_3^2 + n_3^2 &= 1 \end{aligned}$$

**2.025** Transformation from one to another oblique system.

$$\begin{aligned} \cos \widehat{xx'} &= l_1 & \cos \widehat{xy'} &= l_2 & \cos \widehat{xz'} &= l_3 \\ \cos \widehat{yx'} &= m_1 & \cos \widehat{yy'} &= m_2 & \cos \widehat{yz'} &= m_3 \\ \cos \widehat{zx'} &= n_1 & \cos \widehat{zy'} &= n_2 & \cos \widehat{zz'} &= n_3 \end{aligned}$$

$$\Delta = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' \\ y &= m_1 x' + m_2 y' + m_3 z' \\ z &= n_1 x' + n_2 y' + n_3 z' \end{aligned}$$

$$\begin{aligned} \Delta \cdot x' &= (m_2 n_3 - m_3 n_2)x + (n_2 l_3 - n_3 l_2)y + (l_2 m_3 - l_3 m_2)z, \\ \Delta \cdot y' &= (m_3 n_1 - m_1 n_3)x + (n_3 l_1 - n_1 l_3)y + (l_3 m_1 - l_1 m_3)z, \\ \Delta \cdot z' &= (m_1 n_2 - m_2 n_1)x + (n_1 l_2 - n_2 l_1)y + (l_1 m_2 - l_2 m_1)z. \end{aligned}$$

$$\begin{aligned} l_1^2 + m_1^2 + n_1^2 + 2m_1 n_1 \cos \widehat{yz} + 2n_1 l_1 \cos \widehat{zx} + 2l_1 m_1 \cos \widehat{xy} &= 1, \\ l_2^2 + m_2^2 + n_2^2 + 2m_2 n_2 \cos \widehat{yz} + 2n_2 l_2 \cos \widehat{zx} + 2l_2 m_2 \cos \widehat{xy} &= 1, \\ l_3^2 + m_3^2 + n_3^2 + 2m_3 n_3 \cos \widehat{yz} + 2n_3 l_3 \cos \widehat{zx} + 2l_3 m_3 \cos \widehat{xy} &= 1. \end{aligned}$$

$$\begin{aligned} x + y \cos \widehat{xy} + z \cos \widehat{xz} &= l_1 x' + l_2 y' + l_3 z', \\ y + x \cos \widehat{xy} + z \cos \widehat{zy} &= m_1 x' + m_2 y' + m_3 z', \\ z + x \cos \widehat{xz} + y \cos \widehat{zy} &= n_1 x' + n_2 y' + n_3 z'. \end{aligned}$$

**2.026** Transformation from one to another oblique system.

If $n_x$, $n_y$, $n_z$ are the normals to the planes $yz$, $zx$, $xy$ and $n_x'$, $n_y'$, $n_z'$ the normals to the planes $y'z'$, $z'x'$, $x'y'$,

$$\begin{aligned} x\cos\widehat{xn_x} &= x'\cos\widehat{x'n_x} + y'\cos\widehat{y'n_x} + z'\cos\widehat{z'n_x}. \\ y\cos\widehat{yn_y} &= x'\cos\widehat{x'n_y} + y'\cos\widehat{y'n_y} + z'\cos\widehat{z'n_y}. \\ z\cos\widehat{zn_z} &= x'\cos\widehat{x'n_z} + y'\cos\widehat{y'n_z} + z'\cos\widehat{z'n_z}. \end{aligned}$$

$$\begin{aligned} x'\cos\widehat{x'n_x'} &= x\cos\widehat{xn_x'} + y\cos\widehat{yn_x'} + z\cos\widehat{zn_x'}. \\ y'\cos\widehat{y'n_y'} &= x\cos\widehat{xn_y'} + y\cos\widehat{yn_y'} + z\cos\widehat{zn_y'}. \\ z'\cos\widehat{z'n_z'} &= x\cos\widehat{xn_z'} + y\cos\widehat{yn_z'} + z\cos\widehat{zn_z'}. \end{aligned}$$

**2.030** Transformation from rectangular to spherical polar coördinates.

$r$, the radius vector to a point makes an angle $\theta$ with the $z$-axis, the projection of $r$ on the $x$-$y$ plane makes an angle $\phi$ with the $x$-axis.

$$\begin{aligned} x &= r\sin\theta\cos\phi & r^2 &= x^2+y^2+z^2 \\ y &= r\sin\theta\sin\phi & \theta &= \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \\ z &= r\cos\theta & \phi &= \tan^{-1}\frac{y}{x} \end{aligned}$$

**2.031** Transformation from rectangular to cylindrical coördinates.

$\rho$, the perpendicular from the $z$-axis to a point makes an angle $\theta$ with the $x$-$z$ plane.

$$\begin{aligned} x &= \rho\cos\theta & \rho &= \sqrt{x^2+y^2} \\ y &= \rho\sin\theta & \theta &= \tan^{-1}\frac{y}{x} \\ z &= z \end{aligned}$$

**2.032** Curvilinear coördinates in general.

See **4.0**

**2.040** Eulerian Angles.

$Oxyz$ and $Ox'y'z'$ are two systems of rectangular axes with the same origin $O$. $OK$ is perpendicular to the plane $zOz'$ drawn so that if $Oz$ is vertical, and the projection of $Oz'$ perpendicular to $Oz$ is directed to the south, then $OK$ is directed to the east.

Angles $\quad \widehat{z'Oz} = \theta,$

$\widehat{yOK} = \phi,$

$\widehat{y'OK} = \psi.$

The direction cosines of the two systems of axes are given by the following scheme:

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x'$ | $\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi$ | $\sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi$ | $-\sin\theta\cos\psi$ |
| $y'$ | $-\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi$ | $-\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi$ | $\sin\theta\sin\psi$ |
| $z'$ | $\cos\phi\sin\theta$ | $\sin\phi\sin\theta$ | $\cos\theta$ |

**2.050** Trilinear Coördinates.

A point in a plane is defined if its distances from two intersecting lines are given. Let $CA$, $CB$ (Fig. 1) be these lines:

$$PR = p, \quad PS = q, \quad PT = r.$$

Taking $CA$ and $CB$ as the $x$-, $y$-axes, including an angle $C$,

$$x = \frac{p}{\sin C},$$

$$y = \frac{q}{\sin C}.$$

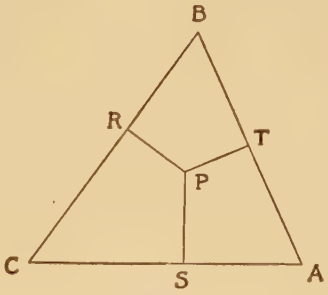

FIG. 1

Any curve $f(x,y) = 0$ becomes:

$$f\left(\frac{p}{\sin C}, \frac{q}{\sin C}\right) = 0.$$

If $s$ is the area of the triangle $CAB$ (triangle of reference),

$$2s = ap + bq + cr,$$
$$a = BC,$$
$$b = CA,$$
$$c = AB,$$

and the equation of a curve may be written in the homogeneous form:

$$f\left(\frac{2sp}{(ap + bq + cr)\sin C}, \frac{2sq}{(ap + bq + cr)\sin C}\right) = 0.$$

**2.060** Quadriplanar Coördinates.

These are the analogue in 3 dimensions of trilinear coördinates in a plane (**2.050**).

$x_1$, $x_2$, $x_3$, $x_4$ denote the distances of a point $P$ from the four sides of a tetrahedron (the tetrahedron of reference); $l_1$, $m_1$, $n_1$; $l_2$, $m_2$, $n_2$; $l_3$, $m_3$, $n_3$; and $l_4$, $m_4$, $n_4$ the direction cosines of the normals to the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ with respect to a rectangular system of coördinates $x$, $y$, $z$; and $d_1$, $d_2$, $d_3$, $d_4$ the distances of these 4 planes from the origin of coördinates:

$$(1)\quad \begin{cases} x_1 = l_1x + m_1y + n_1z - d_1 \\ x_2 = l_2x + m_2y + n_2z - d_2 \\ x_3 = l_3x + m_3y + n_3z - d_3 \\ x_4 = l_4x + m_4y + n_4z - d_4. \end{cases}$$

$s_1$, $s_2$, $s_3$, and $s_4$ are the areas of the 4 faces of the tetrahedron of reference and $V$ its volume:

$$3V = x_1s_1 + x_2s_2 + x_3s_3 + x_4s_4.$$

By means of the first 3 equations of (1) $x$, $y$, $z$ are determined:

$$\begin{aligned} x &= A_1x_1 + B_1x_2 + C_1x_3 + D_1, \\ y &= A_2x_1 + B_2x_2 + C_2x_3 + D_2, \\ z &= A_3x_1 + B_3x_2 + C_3x_3 + D_3. \end{aligned}$$

The equation of any surface,

$$F(x,y,z) = 0,$$

may be written in the homogeneous form:

$$F\left\{\left[A_1x_1 + B_1x_2 + C_1x_3 + \frac{D_1}{3V}(s_1x_1 + s_2x_2 + s_3x_3 + s_4x_4)\right],\right.$$
$$\left[A_2x_1 + B_2x_2 + C_2x_3 + \frac{D_2}{3V}(s_1x_1 + s_2x_2 + s_3x_3 + s_4x_4)\right],$$
$$\left.\left[A_3x_1 + B_3x_2 + C_3x_3 + \frac{D_3}{3V}(s_1x_1 + s_2x_2 + s_3x_3 + s_4x_4)\right]\right\} = 0.$$

## PLANE GEOMETRY

**2.100** The equation of a line:

$$Ax + By + C = 0.$$

**2.101** If $p$ is the perpendicular from the origin upon the line, and $\alpha$ and $\beta$ the angles $p$ makes with the $x$- and $y$-axes:

$$p = x\cos\alpha + y\cos\beta.$$

**2.102** If $\alpha'$ and $\beta'$ are the angles the line makes with the $x$- and $y$-axes:

$$p = y\cos\alpha' - x\cos\beta'.$$

**2.103** The equation of a line may be written

$$y = ax + b.$$

$a$ = tangent of angle the line makes with the $x$-axis,
$b$ = intercept of the $y$-axis by the line.

**2.104** The two lines:

$$y = a_1x + b_1,$$
$$y = a_2x + b_2,$$

intersect at the point:

$$x = \frac{b_2 - b_1}{a_1 - a_2} \qquad y = \frac{a_1b_2 - a_2b_1}{a_1 - a_2}.$$

**2.105** If $\phi$ is the angle between the two lines **2.104**:

$$\tan \phi = \pm \frac{a_1 - a_2}{1 + a_1a_2}.$$

**2.106** Equations of two parallel lines:

$$\begin{cases} Ax + By + C_1 = 0 \\ Ax + By + C_2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = ax + b_1, \\ y = ax + b_2. \end{cases}$$

**2.107** Equations of two perpendicular lines:

$$\begin{cases} Ax + By + C_1 = 0 \\ Bx - Ay + C_2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = ax + b_1, \\ y = -\dfrac{x}{a} + b_2. \end{cases}$$

**2.108** Equation of line through $x_1$, $y_1$ and parallel to the line:

$$Ax + By + C = 0 \quad \text{or} \quad y = ax + b,$$
$$A(x - x_1) + B(y - y_1) = 0 \quad \text{or} \quad y - y_1 = a(x - x_1).$$

**2.109** Equation of line through $x_1$, $y_1$ and perpendicular to the line

$$Ax + By + C = 0 \quad \text{or} \quad y = ax + b,$$
$$B(x - x_1) - A(y - y_1) = 0 \quad \text{or} \quad y - y_1 = -\frac{x - x_1}{a}.$$

**2.110** Equation of line through $x_1$, $y_1$ making an angle $\phi$ with the line $y = ax + b$:

$$y - y_1 = \frac{a + \tan \phi}{1 - a \tan \phi}(x - x_1).$$

**2.111** Equation of line through the two points, $x_1$, $y_1$, and $x_2$, $y_2$:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

**2.112** Perpendicular distance from the point $x_1$, $y_1$ to the line

$$Ax + By + C = 0 \quad \text{or} \quad y = ax + b,$$
$$p = \frac{Ax_1 + By_1 + C}{\sqrt{A_2 + B_2}} \quad \text{or} \quad p = \frac{y_1 - ax_1 - b}{\sqrt{1 + a^2}}.$$

**2.113** Polar equation of the line $y = ax + b$:

$$r = \frac{b \cos \alpha}{\sin (\theta - \alpha)},$$

where

$$\tan \alpha = a.$$

**2.114** If $p$, the perpendicular to the line from the origin, makes an angle $\beta$ with the axis:

$$p = r \cos (\theta - \beta).$$

**2.130** Area of polygon whose vertices are at $x_1, y_1;\ x_2, y_2;\ \ldots\ldots\ldots\ x_n, y_n = A$.

$$2A = y_1(x_n - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_4) + \ldots\ldots + y_n(x_{n-1} - x_1).$$

## PLANE CURVES

**2.200** The equation of a plane curve in rectangular coördinates may be given in the forms:

(a) $y = f(x)$.

(b) $x = f_1(t),\ y = f_2(t)$. The parametric form.

(c) $F(x,y) = 0$.

**2.201** If $\tau$ is the angle between the tangent to the curve and the $x$-axis:

(a) $\tan \tau = \dfrac{dy}{dx} = y'$.

(b) $\tan \tau = \dfrac{\dfrac{df_2(t)}{dt}}{\dfrac{df_1(t)}{dt}}$.

(c) $\tan \tau = -\dfrac{\dfrac{\partial F(x,y)}{\partial x}}{\dfrac{\partial F(x,y)}{\partial y}}$.

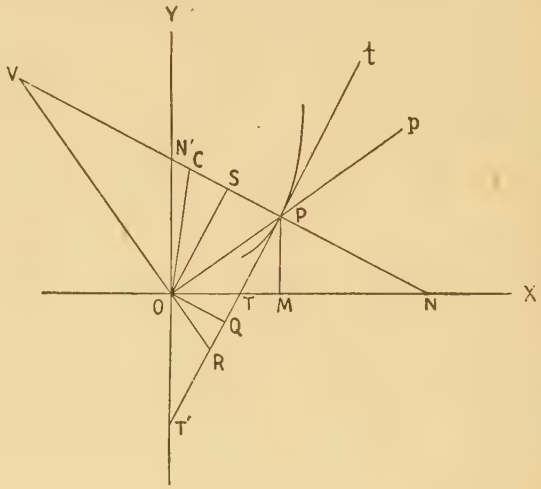


FIG. 2

In the following formulas,

$$y' = \frac{dy}{dx} = \tan \tau \ \textbf{(2.201)}.$$

**2.202** $OM = x$, $MP = y$, angle $XTP = \tau$.

$$TP = y \csc \tau = \frac{y\sqrt{1 + y'^2}}{y'} = \text{tangent},$$

$$TM = y \cot \tau = \frac{y}{y'} = \text{subtangent},$$

$$PN = y \sec \tau = y\sqrt{1 + y'^2} = \text{normal},$$

$$MN = y \tan \tau = yy' = \text{subnormal}.$$

**2.203** $OT = x - \dfrac{y}{y'}$ = intercept of tangent on $x$-axis,

$OT' = y - xy'$ = intercept of tangent on $y$-axis,

$ON = x + yy'$ = intercept of normal on $x$-axis,

$ON' = y + \dfrac{x}{y'}$ = intercept of normal on $y$-axis.

**2.204** $OQ = \dfrac{y - xy'}{\sqrt{1 + y'^2}}$ = distance of tangent from origin = $PS$ = projection of radius vector on normal.

Coördinates of $Q$: $\dfrac{y'(xy' - y)}{1 + y'^2}, \dfrac{y - xy'}{1 + y'^2}$.

**2.205** $OS = \dfrac{x + yy'}{\sqrt{1 + y'^2}}$ = distance of normal from origin = $PQ$ = projection of radius vector on tangent.

Coördinates of $S$: $\dfrac{x + yy'}{1 + y'^2}, \dfrac{(x + yy')y'}{1 + y'^2}$.

**2.206** $OR = \dfrac{\sqrt{x^2 + y^2}\,(y - xy')}{x + yy'}$ = polar subtangent,

$PR = \dfrac{(x^2 + y^2)\sqrt{1 + y'^2}}{x + yy'}$ = polar tangent,

Coördinates of $R$: $\dfrac{y(xy' - y)}{x + yy'}, \dfrac{x(y - xy')}{x + yy'}$.

**2.207** $OV = \dfrac{\sqrt{x^2 + y^2}\,(x + yy')}{y - xy'}$ = polar subnormal,

$PV = \dfrac{(x^2 + y^2)\sqrt{1 + y'^2}}{y - xy'}$ = polar normal,

Coördinates of $V$: $\dfrac{y(x + yy')}{y - xy'}, -\dfrac{x(x + yy')}{y - xy'}$.

**2.210** The equations of the tangent at $x_1, y_1$ to the curve in the three forms of **2.200** are:

(a) $y - y_1 = f'(x_1)\,(x - x_1)$.

(b) $(y - y_1)f_1'(t_1) = (x - x_1)f_2'(t_1)$.

(c) $(x - x_1)\left(\dfrac{\partial F}{\partial x}\right)_{\substack{x=x_1\\y=y_1}} + (y - y_1)\left(\dfrac{\partial F}{\partial y}\right)_{\substack{x=x_1\\y=y_1}} = 0$.

**2.211** The equations of the normal at $x_1, y_1$ to the curve in the three forms of **2.200** are:

(a) $f'(x_1)\,(y - y_1) + (x - x_1) = 0$.

(b) $(y - y_1)f_2'(t_1) + (x - x_1)f_1'(t_1) = 0$.

(c) $(x - x_1)\left(\dfrac{\partial F}{\partial y}\right)_{\substack{x=x_1\\y=y_1}} = (y - y_1)\left(\dfrac{\partial F}{\partial x}\right)_{\substack{x=x_1\\y=y_1}}$.

**2.212** The perpendicular from the origin upon the tangent to the curve $F(x, y) = 0$ at the point $x, y$ is:

$$p = \frac{x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}}.$$

**2.213** Concavity and Convexity. If in the neighborhood of a point $P$ a curve lies entirely on one side of the tangent, it is concave or convex upwards according as $y'' = \frac{d^2y}{dx^2}$ is positive or negative. The positive direction of the axes are shown in figure 2.

**2.220** Convention as to signs. The positive direction of the normal is related to the positive direction of the tangent as the positive $y$-axis is related to the positive $x$-axis. The angle $\tau$ is measured positively in the counter-clockwise direction from the positive $x$-axis to the positive tangent.

**2.221** Radius of curvature $= \rho$; curvature $= 1/\rho$.

$$\frac{1}{\rho} = \frac{d\tau}{ds},$$

where $s$ is the arc drawn from a fixed point of the curve in the direction of the positive tangent.

**2.222** Formulas for the radius of curvature of curves given in the three forms of **2.200.**

(a)
$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}$$

(b)
$$\rho = \frac{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} = \frac{\left(\frac{ds}{dt}\right)^2}{\left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 - \left(\frac{d^2s}{dt^2}\right)^2\right\}^{\frac{1}{2}}}$$

If $s$ is taken as the parameter $t$:

(b′)
$$\frac{1}{\rho} = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}^{\frac{1}{2}}$$

(c)
$$\rho = -\frac{\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right\}^{\frac{3}{2}}}{\frac{\partial^2 F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2 - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2}$$

**2.223** The center of curvature is a point $C$ (fig. 2) on the normal at $P$ such that $PC = \rho$. If $\rho$ is positive $C$ lies on the positive normal (**2.213**); if negative, on the negative normal.

**2.224** The circle of curvature is a circle with $C$ as center and radius $= \rho$.

**2.225** The chord of curvature is the chord of the circle of curvature passing through the origin and the point $P$.

**2.226** The coördinates of the center of curvature at the point $x$, $y$ are $\xi$, $\eta$:

$$\xi = x - \rho \sin \tau \qquad \tan \tau = \frac{dy}{dx}$$

$$\eta = y + \rho \cos \tau$$

If $l'$, $m'$ are the direction cosines of the positive normal,

$$\xi = x + l'\rho$$
$$\eta = y + m'\rho.$$

**2.227** If $l$, $m$ are the direction cosines of the positive tangent and $l'$, $m'$ those of the positive normal,

$$\frac{dl}{ds} = \frac{l'}{\rho}, \quad \frac{dm}{ds} = \frac{m'}{\rho}.$$

$$l' = m, \quad m' = -l,$$

$$\frac{dl'}{ds} = -\frac{l}{\rho}, \quad \frac{dm'}{ds} = -\frac{m}{\rho}$$

**2.228** If the tangent and normal at $P$ are taken as the $x$- and $y$- axes, then

$$\rho = \lim_{x \to 0} \frac{x^2}{2y}$$

**2.229** Points of Inflexion. For a curve given in the form (a) of **2.200** a point of inflexion is a point at which one at least of $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$ exists and is continuous and at which one at least of $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$ vanishes and changes sign.

If the curve is given in the form (b) a point of inflexion, $t_1$, is a point at which the determinant:

$$\begin{vmatrix} f_1''(t_1) & f_2''(t_1) \\ f_1'(t_1) & f_2'(t_1) \end{vmatrix}$$

vanishes and changes sign.

**2.230** Eliminating $x$ and $y$ between the coördinates of the center of curvature (**2.226**) and the corresponding equations of the curve (**2.200**) gives the equation of the evolute of the curve – the locus of the center of curvature. A curve which has a given curve for evolute is called an involute of the given curve.

**2.231** The envelope to a family of curves,

1. $$F(x, y, \alpha) = 0,$$

where $\alpha$ is a parameter, is obtained by eliminating $\alpha$ between (1) and

2. $$\frac{\partial F}{\partial \alpha} = 0.$$

**2.232** If the curve is given in the form,

1. $$x = f_1(t, \alpha)$$

2. $$y = f_2(t, \alpha),$$

the envelope is obtained by eliminating $t$ and $\alpha$ between (1), (2) and the functional determinant,

3. $$\frac{\partial(f_1, f_2)}{\partial(t, \alpha)} = 0 \quad \text{(see 1.370)}$$

**2.233** Pedal Curves. The locus of the foot of the perpendicular from a fixed point upon the tangent to a given curve is the pedal of the given curve with reference to the fixed point.

**2.240** Asymptotes. The line

$$y = ax + b$$

is an asymptote to the curve $y = f(x)$ if

$$a = \lim_{x \to \infty} f'(x)$$

$$b = \lim_{x \to \infty} [f(x) - xf'(x)]$$

**2.241** If the curve is

$$x = f_1(t), \; y = f_2(t),$$

and if for a value of $t$, $t_1$, $f_1$ or $f_2$ becomes infinite, there will be an asymptote if for that value of $t$ the direction of the tangent to the curve approaches a limit and the distance of the tangent from a fixed point approaches a limit.

**2.242** An asymptote may sometimes be determined by expanding the equation of the curve in a series,

$$y = \sum_{k=0}^{n} a_k x^k + \sum_{k=1}^{\infty} \frac{b_k}{x^k}.$$

If

$$\lim_{x \to \infty} \sum_{k=1}^{\infty} \frac{b_k}{x^k} = 0,$$

the equation of the asymptote is

$$y = \sum_{k=0}^{n} a_k x^k$$

If of the first degree in $x$, this represents a rectilinear asymptote; if of a higher degree, a curvilinear asymptote.

**2.250** Singular Points. If the equation of the curve is $F(x, y) = 0$, singular points are those for which

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0.$$

Put,

$$\Delta = \frac{\partial^2 F}{\partial x^2}\,\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2$$

If $\Delta < 0$ the singular point is a double point with two distinct tangents.

$\Delta > 0$ the singular point is an isolated point with no real branch of the curve through it.

$\Delta = 0$ the singular point is an osculating point, or a cusp. The curve has two branches, with a common tangent, which meet at the singular point.

If $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial x\,\partial y}$ simultaneously vanish at a point the singular point is one of higher order.

## PLANE CURVES, POLAR COÖRDINATES

**2.270** The equation of the curve is given in the form,

$$r = f(\theta).$$

In figure 2, $OP = r$, angle $XOP = \theta$, angle $XTP = \tau$, angle $pPt = \phi$.

**2.271** $\theta$ is measured in the counter-clockwise direction from the initial line, $OX$, and $s$, the arc, is so chosen as to increase with $\theta$. The angle $\phi$ is measured in the counter-clockwise direction from the positive radius vector to the positive tangent. Then,

$$\tau = \theta + \phi.$$

**2.272**

$$\tan \phi = \frac{r\,d\theta}{dr}$$

$$\sin \phi = \frac{r\,d\theta}{ds}$$

$$\cos \phi = \frac{dr}{ds}$$

**2.273**

$$\tan \tau = \frac{\sin\theta \frac{dr}{d\theta} + r \cos\theta}{\cos\theta \frac{dr}{d\theta} - r \sin\theta}$$

$$ds = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta$$

**2.274**

$$PR = r\sqrt{1 + \left(\frac{rd\theta}{dr}\right)^2} = \text{polar tangent}$$

$$PV = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \text{polar normal}$$

$$OR = r^2\frac{d\theta}{dr} = \text{polar subtangent}$$

$$OV = \frac{dr}{d\theta} = \text{polar subnormal.}$$

**2.275**

$$OQ = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = p = \text{distance of tangent from origin.}$$

$$OS = \frac{r\frac{dr}{d\theta}}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = \text{distance of normal from origin.}$$

**2.276** If $u = \frac{1}{r}$, the curve $r = f(\theta)$ is concave or convex to the origin according as

$$u + \frac{d^2u}{d\theta^2}$$

is positive or negative. At a point of inflexion this quantity vanishes and changes sign.

**2.280** The radius of curvature is,

$$\rho = \frac{\left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

**2.281** If $u = \frac{1}{r}$ the radius of curvature is

$$\rho = \frac{\left\{ u^2 + \left(\frac{du}{d\theta}\right)^2 \right\}^{\frac{3}{2}}}{u^3\left(u + \frac{d^2u}{d\theta^2}\right)}.$$

**2.282** If the equation of the curve is given in the form,

$$r = f(s)$$

where $s$ is the arc measured from a fixed point of the curve,

$$\rho = \frac{r\sqrt{1 - \left(\frac{dr}{ds}\right)^2}}{r\frac{d^2r}{ds^2} + \left(\frac{dr}{ds}\right)^2 - 1}.$$

**2.283** If $p$ is the perpendicular from the origin upon the tangent to the curve,

1. $\rho = r\frac{dr}{dp}$ 2. $\rho = p + \frac{d^2p}{d\tau^2}$

**2.284** If $u = \frac{1}{r}$

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

**2.285**

$$\frac{d^2u}{d\theta^2} + u = \frac{r^2}{p^3}\left(\frac{dp}{dr}\right)$$

**2.286** Polar coördinates of the center of curvature, $r_1$, $\theta_1$:

$$r_1^2 = \frac{r^2\left\{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}^2 + \left(\frac{dr}{d\theta}\right)^2\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}^2}{\left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}^2}.$$

$$\theta_1 = \theta + \chi,$$

$$\tan\chi = \frac{\left(\frac{dr}{d\theta}\right)^3 + r^2\frac{dr}{d\theta}}{r\left(\frac{dr}{d\theta}\right)^2 - r^2\frac{d^2r}{d\theta^2}}.$$

**2.287** If $2c$ is the chord of curvature (**2.225**):

$$2c = 2p\frac{dr}{dp} = 2\rho\frac{p}{r},$$

$$= 2\frac{u^2 + \left(\frac{du}{d\theta}\right)^2}{u^2\left(u + \frac{d^2u}{d\theta^2}\right)}.$$

**2.290** Rectilinear Asymptotes. If $r$ approaches $\infty$ as $\theta$ approaches an angle $\alpha$, and if $r(\alpha - \theta)$ approaches a limit, $b$, then the straight line

$$r\sin(\alpha - \theta) = b$$

is an asymptote to the curve $r = f(\theta)$.

**2.295** Intrinsic Equation of a plane curve. An intrinsic equation of a plane curve is one giving the radius of curvature, $\rho$, as a function of the arc, $s$,

$$\rho = f(s)$$

If $\tau$ is the angle between the $x$-axis and the positive tangent (**2.271**):

$$d\tau = \frac{ds}{f(s)} \qquad x = x_0 + \int_{s_0}^{s} \cos\,\tau \cdot ds$$

$$\tau = \tau_0 + \int_{s_0}^{s} \frac{ds}{f(s)} \qquad y = y_0 + \int_{s_0}^{s} \sin\,\tau \cdot ds.$$

**2.300** The general equation of the second degree:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; \qquad a_{hk} = a_{kh}$$

$A_{hk}$ = Minor of $a_{hk}$.

Criterion giving the nature of the curve:

<table>
<tr><td></td><td colspan="3">$A_{33} \neq 0$</td><td colspan="2">$A_{33} = 0$</td></tr>
<tr><td rowspan="3">$A \neq 0$</td><td>$A_{33} < 0$</td><td colspan="2">$A_{33} > 0$</td><td colspan="2" rowspan="3">Parabola</td></tr>
<tr><td rowspan="2">Hyperbola</td><td colspan="2">$a_{11}A$ or $a_{22}A$</td></tr>
<tr><td>$<0$</td><td>$>0$</td></tr>
<tr><td></td><td></td><td>Ellipse</td><td>Imaginary Curve</td><td></td></tr>
<tr><td rowspan="3">$A = 0$</td><td>$A_{33} < 0$</td><td colspan="2">$A_{33} > 0$</td><td>$A_{11}$ or $A_{22}$ $<0$ | $>0$</td><td>$A_{11} = A_{22} = 0$</td></tr>
<tr><td>Pair of Real Straight Lines</td><td colspan="2">Pair of Imaginary Lines</td><td>Real | Imaginary<br>Pair of Parallel Lines</td><td rowspan="2">Double Line</td></tr>
<tr><td colspan="3">Intersection Finite</td><td></td></tr>
</table>

(Pascal: Repertorium der höheren Mathematik, II, 1, p. 228)

**2.400** Parabola (Fig. 3).

**2.401** $O$, Vertex; $F$, Focus; ordinate through $D$, Directrix.

Equation of parabola, origin at $O$,

$$y^2 = 4ax$$

$x = OM$, $y = MP$,

$OF = OD = a$

$FL = 2a$ = semi latus rectum.

$FP = D'P$.

**2.402** $FP = FT = MD = x + a$.

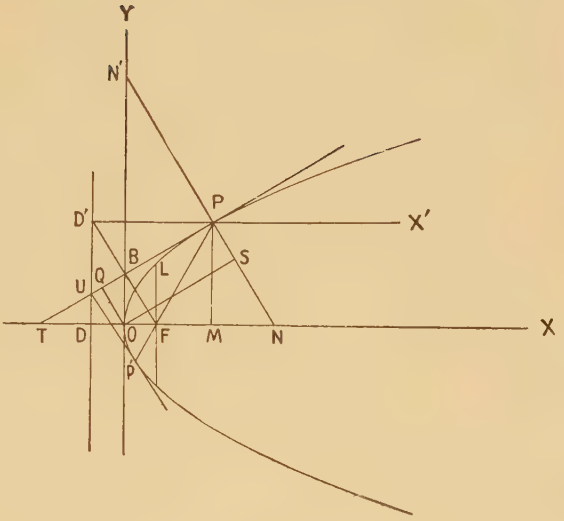


FIG. 3

$$NP = 2\sqrt{a(a+x)},\ TM = 2x,\ MN = 2a,\ ON = x + 2a.$$

$$ON' = \sqrt{\frac{x}{a}}(x + 2a),\ OQ = x\sqrt{\frac{a}{a+x}},\ OS = (x + 2a)\sqrt{\frac{x}{a+x}}.$$

$FB$ perpendicular to tangent $TP$.

$$FB = \sqrt{a(a+x)},\ TP = 2TB = 2\sqrt{x(a+x)}.$$

$$\overline{FB}^2 = FT \times FO = FP \times FO.$$

The tangents $TP$ and $UP'$ at the extremities of a focal chord $PFP'$ meet on the directrix at $U$ at right angles.

$\tau$ = angle $XTP$.

$$\tan \tau = \sqrt{\frac{a}{x}}.$$

The tangent at $P$ bisects the angles $FPD'$ and $FUD'$.

**2.403** Radius of curvature:

$$\rho = \frac{2(x+a)^{\frac{3}{2}}}{\sqrt{a}} = \frac{1}{4}\frac{\overline{NP}^3}{a^2}.$$

Coördinates of center of curvature:

$$\xi = 3x + 2a,\ \eta = -2x\sqrt{\frac{x}{a}}.$$

Equation of Evolute:

$$27ay^2 = 4(x - 2a)^3.$$

**2.404** Length of arc of parabola measured from vertex,

$$s = \sqrt{x(x+a)} + a \log \left( \sqrt{1 + \frac{x}{a}} + \sqrt{\frac{x}{a}} \right).$$

Area $OPMO = \frac{1}{3} xy$.

**2.405** Polar equation of parabola:

$$r = FP,$$
$$\theta = \text{angle } XFP,$$
$$r = \frac{2a}{1 - \cos \theta}.$$

**2.406** Equation of Parabola in terms of $p$, the perpendicular from $F$ upon the tangent, and $r$, the radius vector $FP$:

$$\frac{l}{p^2} = \frac{2}{r}$$

$l$ = semi latus rectum.

**2.410** Ellipse (Fig. 4).

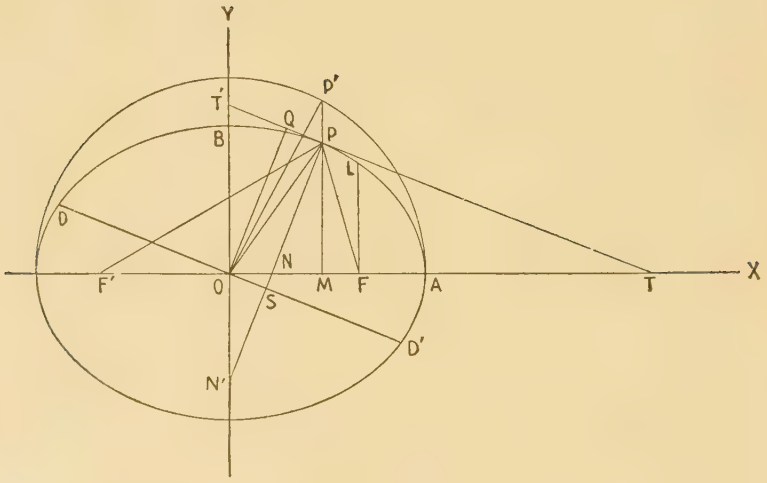


FIG. 4

**2.411** $O$, Centre; $F$, $F'$, Foci.

Equation of Ellipse origin at $O$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = OM$, $y = MP$, $a = OA$, $b = OB$.

**2.412** Parametric Equations of Ellipse,

$$x = a \cos \phi, \qquad y = b \sin \phi.$$

$\phi$ = angle $XOP'$, where $P'$ is the point where the ordinate at $P$ meets the eccentric circle, drawn with $O$ as center and radius $a$.

**2.413** $OF = OF' = ea$

$$e = \text{eccentricity} = \frac{\sqrt{a^2 - b^2}}{a},$$

$$FL = \frac{b^2}{a} = a(1 - e^2) = \text{semi latus rectum.}$$

$$F'P = a + ex, \quad FP = a - ex, \quad FP + F'P = 2a.$$

$\tau$ = angle $XTT'$.

$$\tan \tau = -\frac{bx}{a\sqrt{a^2 - x^2}}.$$

$$NM = \frac{b^2x}{a^2}, \quad ON = e^2x, \quad OT = \frac{a^2}{x}, \quad OT' = \frac{b^2}{y}, \quad MT = \frac{a^2 - x^2}{x},$$

$$PT = \frac{\sqrt{a^2 - x^2}\sqrt{a^2 - e^2x^2}}{x}, \quad ON' = \frac{e^2a}{b}\sqrt{a^2 - x^2}, \quad PS = \frac{ab}{\sqrt{a^2 - e^2x^2}},$$

$$OS = \frac{e^2x\sqrt{a^2 - x^2}}{\sqrt{a^2 - e^2x^2}}.$$

**2.414** $DD'$ parallel to $T'T$; $DD'$ and $PP'$ are conjugate diameters:

$$OD^2 = a^2 - e^2x^2 = FP \times F'P.$$

$$OP^2 + OD^2 = a^2 + b^2.$$

$$PS \times OD = ab.$$

Equation of Ellipse referred to conjugate diameters as axes:

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$$

$\alpha$ = angle $XOP$

$\beta$ = angle $XOD$

$a' = OD'$ $\qquad a'^2 = \dfrac{a^2b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$ $\qquad \tan \alpha \tan \beta = -\dfrac{b^2}{a^2}$

$b' = OP$ $\qquad b'^2 = \dfrac{a^2b^2}{a^2 \sin^2 \beta + b^2 \cos^2 \beta}$

**2.415** Radius of curvature of Ellipse:

$$\rho = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4} = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}.$$

$$\text{angle } FPN = \text{angle } F'PN = \omega,$$

$$\tan \omega = \frac{eay}{b^2},$$

$$\frac{2}{\rho \cos \omega} = \frac{1}{FP} + \frac{1}{F'P}.$$

Coördinates of center of curvature:

$$\xi = \frac{e^2x^3}{a^2},\ \eta = -\frac{a^2e^2y^3}{b^4}.$$

Equation of Evolute of Ellipse,

$$\left(\frac{ax}{e^2}\right)^{\frac{2}{3}} + \left(\frac{by}{e^2}\right)^{\frac{2}{3}} = 1.$$

**2.416** Area of Ellipse, $\pi ab$.

Length of arc of Ellipse,

$$s = a\int_0^{\phi} \sqrt{1 - e^2 \sin^2\phi}\, d\phi.$$

**2.417** Polar Equation of Ellipse,

$$r = F'P,\ \theta = \text{angle } XF'P,$$

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$$

**2.418**

$$r = OP,\ \theta = \text{angle } XOP,$$

$$r = \frac{b}{\sqrt{1 - e^2\cos^2\theta}}$$

**2.419** Equation of Ellipse in terms of $p$, the perpendicular from $F$ upon the tangent at $P$, and $r$, the radius vector $FP$:

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a}.$$

$l$ = semi latus rectum.

**2.420** Hyperbola (Fig. 5).

**2.421** $O$, Center; $F$, $F'$, Foci.

Equation of hyperbola, origin at $O$,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x = OM,\ y = MP,\ a = OA = OA'.$$

**2.422** Parametric Equations of hyperbola,

$$x = a\cosh u,\ y = b\sinh u.$$

or

$$x = a\sec\phi,\ \ y = b\tan\phi.$$

$\phi$ = angle $XOP'$, where $P'$ is the point where the ordinate at $T$ meets the circle of radius $a$, center $O$.

**2.423** $OF = OF' = ea.$

$e$ = eccentricity = $\dfrac{\sqrt{a^2 + b^2}}{a}$.

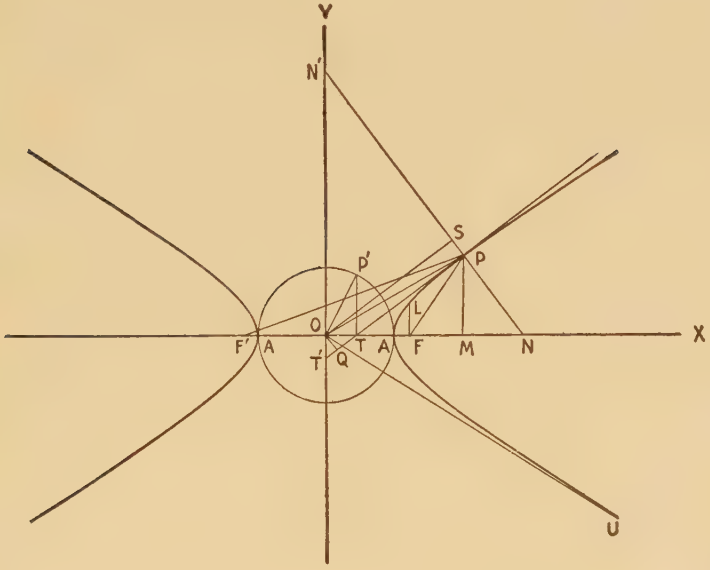


FIG. 5

$FL = \dfrac{b^2}{a} = a(e^2 - 1)$ = semi latus rectum.

$F'P = ex + a,\ FP = ex - a,\ F'P - FP = 2a.$

$\tau$ = angle $XTP$.

$$\tan \tau = \frac{bx}{a\sqrt{x^2 - a^2}}.$$

$$NM = \frac{b^2x}{a^2},\ ON = e^2x,\ OT = \frac{a^2}{x},\ OT' = \frac{b^2}{y},$$

$$MT = \frac{x^2 - a^2}{x},\ PT = \frac{\sqrt{x^2 - a^2}\sqrt{e^2x^2 - a^2}}{x},\ ON' = \frac{e^2a}{b}\sqrt{x^2 - a^2},$$

$$PS = \frac{ab}{\sqrt{e^2x^2 - a^2}},\ OS = \frac{e^2x\sqrt{x^2 - a^2}}{\sqrt{e^2x^2 - a^2}}.$$

**2.424** $OU$ = Asymptote.

$$\tan XOU = \frac{b}{a}.$$

$b$ = distance of vertex $A$ from asymptote.

**2.425** Radius of curvature of hyperbola,

$$\rho = \frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}.$$

$$\text{angle } F'PT = \text{angle } FPT.$$

$$\text{angle } FPN = \omega = \frac{\pi}{2} - FPT.$$

$$\text{angle } F'PN = \omega' = \frac{\pi}{2} + F'PT.$$

$$\tan\ \omega = \frac{aey}{b^2}.$$

$$\cos\ \omega = \frac{b}{\sqrt{e^2x^2 - a^2}}$$

$$\frac{2}{\rho \cos\ \omega} = \frac{1}{FP} - \frac{1}{F'P}.$$

Coördinates of center of curvature,

$$\xi = \frac{e^2x^3}{a^2},\ \eta = -\frac{a^2e^2y^3}{b^4}.$$

Equation of Evolute of hyperbola,

$$\left(\frac{ax}{e^2}\right)^{\frac{2}{3}} - \left(\frac{by}{e^2}\right)^{\frac{2}{3}} = 1.$$

**2.426** In a rectangular hyperbola $b = a$; the asymptotes are perpendicular to each other. Equation of rectangular hyperbola with asymptotes as axes and origin at $O$:

$$xy = \frac{a^2}{2}.$$

**2.427** Length of arc of hyperbola,

$$s = \frac{b^2}{ae}\int_0^{\phi} \frac{\sec^2 \phi\ d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},\quad k = \frac{1}{e},\quad \tan \phi = \frac{aey}{b^2}.$$

**2.428** Polar Equation of hyperbola:

$$r = F'P,\quad \theta = XF'P,\quad r = a\,\frac{e^2 - 1}{e \cos\ \theta - 1},$$

$$r = OP,\quad \theta = XOP,\quad r^2 = \frac{b^2}{e^2 \cos^2\ \theta - 1}.$$

**2.429** Equation of right-hand branch of hyperbola in terms of $p$, the perpendicular from $F$ upon the tangent at $P$ and $r$, the radius vector $FP$,

$$\frac{l}{p^2} = \frac{2}{r} + \frac{1}{a}.$$

$l$ = semi latus rectum.

**2.450** Cycloids and Trochoids.

If a circle of radius $a$ rolls on a straight line as base the extremity of any radius, $a$, describes a cycloid. The rectangular equation of a cycloid is:

$$x = a(\phi - \sin \phi),$$
$$y = a(1 - \cos \phi),$$

where the $x$-axis is the base with the origin at the initial point of contact. $\phi$ is the angle turned through by the moving circle. (Fig. 6.)

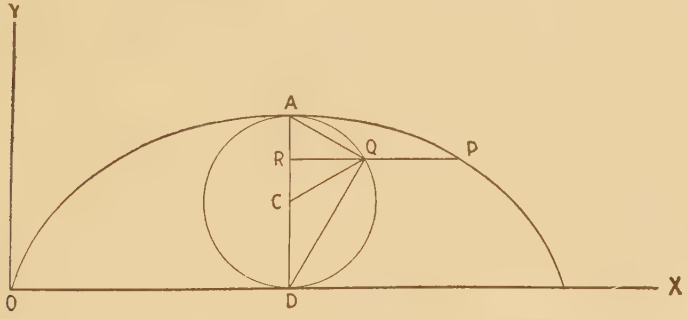


FIG. 6

$A$ = vertex of cycloid.

$C$ = center of generating circle, drawn tangent at $A$.

The tangent to the cycloid at $P$ is parallel to the chord $AQ$.

Arc $AP$ = 2 × chord $AQ$.

The radius of curvature at $P$ is parallel to the chord $QD$ and equal to 2 × chord $QD$.

$PQ$ = circular arc $AQ$.

Length of cycloid: $s = 8a$; $a = CA$.

Area of cycloid: $S = 3\pi a^2$.

**2.451** A point on the radius, $b > a$, describes a prolate trochoid. A point, $b < a$, describes a curtate trochoid. The general equation of trochoids and cycloids is

$$x = a\phi - (a + d) \sin \phi,$$
$$y = (a + d)(1 - \cos \phi),$$
$$d = 0 \text{ Cycloid},$$
$$d > 0 \text{ Prolate trochoid},$$
$$d < 0 \text{ Curtate trochoid}.$$

Radius of curvature:

$$\rho = \frac{(2ay + d^2)^{\frac{3}{2}}}{ay + ad + d^2}.$$

**2.452** Epi- and Hypocycloids. An epicycloid is described by a point on a circle of radius $a$ that rolls on the convex side o a fixed circle of radius $b$. An hypocycloid is described by a point on a circle of radius $a$ that rolls on the concave side of a fixed circle of radius $b$.

Equations of epi- and hypocycloids.

Upper sign: Epicycloid,

Lower sign: Hypocycloid.

$$x = (b \pm a) \cos \phi \mp \cos \frac{b \pm a}{a} \phi,$$

$$y = (b \pm a) \sin \phi - a \sin \frac{b \pm a}{a} \phi.$$

The origin is at the center of the fixed circle. The $x$-axis is the line joining the centers of the two circles in the initial position and $\phi$ is the angle turned through by the moving circle.

Radius of curvature:

$$\rho = \frac{2a(b \pm a)}{b \pm 2a} \sin \frac{a}{2b} \phi.$$

**2.453** In the epicycloid put $b = a$. The curve becomes a Cardioid:

$$(x^2 + y^2)^2 - 6a^2(x^2 + y^2) + 8a^3x = 3a^4.$$

**2.454** Catenary. The equation may be written:

1. $$y = \frac{1}{2} a(e^{\frac{x}{a}} + e^{-\frac{x}{a}}).$$

2. $$y = a \cosh \frac{x}{a}.$$

3. $$x = a \log \frac{y \pm \sqrt{y^2 - a^2}}{a}.$$

The radius of curvature, which is equal to the length of the normal, is:

$$\rho = a \cosh^2 \frac{x}{a}.$$

**2.455** Spiral of Archimedes. A point moving uniformly along a line which rotates uniformly about a fixed point describes a spiral of Archimedes. The equation is:

$$r = a\theta,$$

or

$$\sqrt{x^2 + y^2} = a \tan^{-1} \frac{y}{x}.$$

The polar subtangent = polar subnormal = $a$.

Radius of curvature:

$$\rho = \frac{r(1 + \theta^2)^{\frac{3}{2}}}{\theta(2 + \theta^2)} = \frac{(r^2 + a^2)^{\frac{3}{2}}}{r^2 + 2a^2}.$$

**2.456** Hyperbolic spiral:

$$r\theta = a.$$

**2.457** Parabolic spiral:

$$r^2 = a^2\theta.$$

**2.458** Logarithmic or equiangular spiral:

$r = ae^{n\theta}$,
$n = \cot \alpha = \text{const.}$,
$\alpha$ = angle tangent to curve makes with the radius vector.

**2.459** Lituus:

$$r\sqrt{\theta} = a.$$

**2.460** Neoid:

$$r = a + b\theta.$$

**2.461** Cissoid:

$$(x^2 + y^2)x = 2ay^2,$$
$$r = 2a \tan \theta \sin \theta.$$

**2.462** Cassinoid:

$$(x^2 + y^2 + a^2)^2 = 4a^2x^2 + b^4,$$
$$r^4 - 2a^2r^2 \cos 2\theta = b^4 - a^4.$$

**2.463** Lemniscate ($b = a$ in Cassinoid):

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2),$$
$$r^2 = 2a^2 \cos 2\theta.$$

**2.464** Conchoid:

$$x^2y^2 = (b + y)^2(a^2 - y^2).$$

**2.465** Witch of Agnesi:

$$x^2y = 4a^2(2a - y).$$

**2.466** Tractrix:

$$x = \tfrac{1}{2}a \log \frac{a + \sqrt{a^2 - y^2}}{a - \sqrt{a^2 - y^2}} - \sqrt{a^2 - y^2},$$

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}},$$

$$\rho = \frac{a\sqrt{a^2 - y^2}}{y}.$$

## SOLID GEOMETRY

**2.600** The Plane. The general equation of the plane is:

$$Ax + By + Cz + D = 0.$$

**2.601** $l$, $m$, $n$ are the direction cosines of the normal to the plane and $p$ is the perpendicular distance from the origin upon the plane.

$$l,\ m,\ n = \frac{A,\ B,\ C}{\sqrt{A^2 + B^2 + C^2}},$$

$$p = lx + my + nz,$$

$$p = -\frac{D}{\sqrt{A^2 + B^2 + C^2}}.$$

**2.602** The perpendicular from the point $x_1, y_1, z_1$ upon the plane $Ax + By + Cz + D = 0$ is:

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}.$$

**2.603** $\theta$ is the angle between the two planes:

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

**2.604** Equation of the plane passing through the three points $(x_1, y_1, z_1)$ $(x_2, y_2, z_2)$ $(x_3, y_3, z_3)$:

$$x \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} + y \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} + z \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

## THE RIGHT LINE

**2.620** The equations of a right line passing through the point $x_1, y_1, z_1$, and whose direction cosines are $l, m, n$ are:

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

**2.621** $\theta$ is the angle between the two lines whose direction cosines are $l_1, m_1, n_1$ and $l_2, m_2, n_2$:

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2,$$
$$\sin^2\theta = (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2.$$

**2.622** The direction cosines of the normal to the plane defined by the two lines whose direction cosines are $l_1, m_1, n_1$ and $l_2, m_2n_2$ are:

$$\frac{m_1n_2 - m_2n_1}{\sin\theta}, \quad \frac{n_1l_2 - n_2l_1}{\sin\theta}, \quad \frac{l_1m_2 - l_2m_1}{\sin\theta}.$$

**2.623** The shortest distance between the two lines:

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \quad \text{and} \quad \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2},$$

is:

$$d = \frac{(x_1 - x_2)(m_1n_2 - m_2n_1) + (y_1 - y_2)(n_1l_2 - n_2l_1) + (z_1 - z_2)(l_1m_2 - l_2m_1)}{\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}^{\frac{1}{2}}},$$

**2.624** The direction cosines of the shortest distance between the two lines are:

$$\frac{(m_1n_2 - n_2m_1),\ (n_1l_2 - n_2l_1),\ (l_1m_2 - l_2m_1)}{\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}^{\frac{1}{2}}}.$$

**2.625** The perpendicular distance from the point $x_2, y_2, z_2$ to the line:

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

is:

$$d = \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}^{\frac{1}{2}} - \{l_1(x_2 - x_1) + m_1(y_2 - y_1) + n_1(z_2 - z_1)\}.$$

**2.626** The direction cosines of the line passing through the two points $x_1, y_1, z_1$ and $x_2, y_2, z_2$ are:

$$\frac{(x_2 - x_1), \quad (y_2 - y_1), \quad (z_2 - z_1)}{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}^{\frac{1}{2}}}.$$

**2.627** The two lines:

$$x = m_1 z + p_1, \qquad x = m_2 z + p_2,$$

and

$$y = n_1 z + q_1, \qquad y = n_2 z + q_2,$$

intersect at a point if,

$$(m_1 - m_2)(q_1 - q_2) - (n_1 - n_2)(p_1 - p_2) = 0.$$

The coördinates of the point of intersection are:

$$x = \frac{m_1 p_2 - m_2 p_1}{m_1 - m_2}, \quad y = \frac{n_1 q_2 - n_2 q_1}{n_1 - n_2}, \quad z = \frac{p_2 - p_1}{m_1 - m_2} = \frac{q_2 - q_1}{n_1 - n_2}.$$

The equation of the plane containing the two lines is then

$$(n_1 - n_2)(x - m_1 z - p_1) = (m_1 - m_2)(y - n_1 z - q_1).$$

## SURFACES

**2.640** A single equation in $x, y, z$ represents a surface:

$$F(x, y, z) = 0.$$

**2.641** The direction cosines of the normal to the surface are:

$$l, m, n = \frac{\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial z}}{\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right\}^{\frac{1}{2}}}.$$

**2.642** The perpendicular from the origin upon the tangent plane at $x, y, z$ is:

$$p = lx + my + nz.$$

**2.643** The two principal radii of curvature of the surface $F(x, y, z) = 0$ are given by the two roots of:

$$\begin{vmatrix} \frac{k}{\rho}+\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x\partial y} & \frac{\partial^2 F}{\partial x\partial z} & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial x\partial y} & \frac{k}{\rho}+\frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y\partial z} & \frac{\partial F}{\partial y} \\ \frac{\partial^2 F}{\partial x\partial z} & \frac{\partial^2 F}{\partial y\partial z} & \frac{k}{\rho}+\frac{\partial^2 F}{\partial z^2} & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & 0 \end{vmatrix} = 0,$$

where:

$$k^2 = \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2.$$

**2.644** The coördinates of each center of curvature are:

$$\xi = x + \frac{\rho}{k}\frac{\partial F}{\partial x}, \qquad \eta = y + \frac{\rho}{k}\frac{\partial F}{\partial y}, \qquad \zeta = z + \frac{\rho}{k}\frac{\partial F}{\partial z}.$$

**2.645** The envelope of a family of surfaces:

1. $$F(x, y, z, \alpha) = 0$$

is found by eliminating $\alpha$ between (1) and

2. $$\frac{\partial F}{\partial \alpha} = 0.$$

**2.646** The characteristic of a surface is a curve defined by the two equations (1) and (2) in **2.645**.

**2.647** The envelope of a family of surfaces with two variable parameters, $\alpha$, $\beta$, is obtained by eliminating $\alpha$ and $\beta$ between:

1. $$F(x, y, z, \alpha, \beta) = 0.$$

2. $$\frac{\partial F}{\partial \alpha} = 0.$$

3. $$\frac{\partial F}{\partial \beta} = 0.$$

**2.648** The equations of a surface may be given in the parametric form:

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v).$$

The equation of a tangent plane at $x_1, y_1, z_1$ is:

$$(x - x_1)\frac{\partial(f_2, f_3)}{\partial(u, v)} + (y - y_1)\frac{\partial(f_3, f_1)}{\partial(u, v)} + (z - z_1)\frac{\partial(f_1, f_2)}{\partial(u, v)} = 0,$$

where

$$\frac{\partial(f_2, f_3)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{vmatrix}, \text{ etc. See } \mathbf{1.370}.$$

**2.649** The direction cosines to the normal to the surface in the form **2.648** are:

$$l,\ m,\ n = \frac{\dfrac{\partial(f_2,\ f_3)}{\partial(u,\ v)},\ \dfrac{\partial(f_3,\ f_1)}{\partial(u,\ v)},\ \dfrac{\partial(f_1,\ f_2)}{\partial(u,\ v)}}{\left\{\left(\dfrac{\partial(f_2,\ f_3)}{\partial(u,\ v)}\right)^2+\left(\dfrac{\partial(f_3,\ f_1)}{\partial(u,\ v)}\right)^2+\left(\dfrac{\partial(f_1,\ f_2)}{\partial(u,\ v)}\right)^2\right\}^{\frac{1}{2}}}.$$

**2.650** If the equation of the surface is:

$$z = f(x,\ y),$$

the equation of the tangent plane at $x_1,\ y_1,\ z_1$ is:

$$z - z_1 = \left(\frac{\partial f}{\partial x}\right)_1 (x - x_1) + \left(\frac{\partial f}{\partial y}\right)_1 (y - y_1).$$

**2.651** The direction cosines of the normal to the surface in the form **2.650** are:

$$l,\ m,\ n = \frac{-\left(\dfrac{\partial f}{\partial x}\right),\ -\left(\dfrac{\partial f}{\partial y}\right),\ +1}{\left\{1+\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2\right\}^{\frac{1}{2}}}.$$

**2.652** The two principal radii of curvature of the surface in the form **2.650** are given by the two roots of:

$$(rt - s^2)\rho^2 - \{(1+q^2)r - 2pqs + (1+p^2)t\}\sqrt{1+p^2+q^2}\,\rho + (1+p^2+q^2)^2 = 0,$$

where

$$p = \frac{\partial f}{\partial x},\quad q = \frac{\partial f}{\partial y},\quad r = \frac{\partial^2 f}{\partial x^2},\quad s = \frac{\partial^2 f}{\partial x \partial y},\quad t = \frac{\partial^2 f}{\partial y^2}.$$

**2.653** If $\rho_1$ and $\rho_2$ are the two principal radii of curvature of a surface, and $\rho$ is the radius of curvature in a plane making an angle $\phi$ with the plane of $\rho_1$,

$$\frac{1}{\rho} = \frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2}.$$

**2.654** If $\rho$ and $\rho'$ are the radii of curvature in any two mutually perpendicular planes, and $\rho_1$ and $\rho_2$ the two principal radii of curvature:

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

**2.655** Gauss's measure of the curvature of a surface is:

$$\frac{1}{\rho} = \frac{1}{\rho_1\rho_2}.$$

## SPACE CURVES

**2.670** The equations of a space curve may be given in the forms:

(a) $F_1(x,\ y,\ z) = 0,\quad F_2(x,\ y,\ z) = 0.$

(b) $x = f_1(t),\quad y = f_2(t),\quad z = f_3(t).$

(c) $y = \phi(x),\quad z = \psi(x).$

**2.671** The direction cosines of the tangent to a space curve in the form (a) are:

$$l = \frac{\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}}{T},$$

$$m = \frac{\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}}{T},$$

$$n = \frac{\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}}{T},$$

where $T$ is the positive root of:

$$T^2 = \left\{\left(\frac{\partial F_1}{\partial x}\right)^2 + \left(\frac{\partial F_1}{\partial y}\right)^2 + \left(\frac{\partial F_1}{\partial z}\right)^2\right\}\left\{\left(\frac{\partial F_2}{\partial x}\right)^2 + \left(\frac{\partial F_2}{\partial y}\right)^2 + \left(\frac{\partial F_2}{\partial z}\right)^2\right\} - \left\{\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}\right\}^2.$$

**2.672** The direction cosines of the tangent to a space curve in the form (b) are:

$$l,\ m,\ n = \frac{x',\ y',\ z'}{\{x'^2 + y'^2 + z'^2\}^{\frac{1}{2}}},$$

where the accents denote differentials with respect to $t$.

**2.673** If $s$, the length of arc measured from a fixed point on the curve is the parameter, $t$:

$$l,\ m,\ n = \frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds}.$$

**2.674** The principal radius of curvature of a space curve in the form (b) is:

$$\rho = \frac{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}}{\{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2\}^{\frac{1}{2}}} = \frac{s'^2}{(x''^2 + y''^2 + z''^2 - s''^2)^{\frac{1}{2}}},$$

where the double accents denote second differentials with respect to $t$, and $s$, the length of arc, is a function of $t$.

**2.675** When $t = s$:

$$\frac{1}{\rho} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}}$$

**2.676** The direction cosines of the principal normal to the space curve in the form (b) are:

$$l' = \frac{z'(z'x'' - x'z'') - y'(x'y'' - y'x'')}{L},$$

$$m' = \frac{x'(x'y'' - y'x'') - z'(y'z'' - z'y'')}{L},$$

$$n' = \frac{y'(y'z'' - z'y'') - x'(z'x'' - x'z'')}{L},$$

where

$$L = \{x'^2 + y'^2 + z'^2\}^{\frac{1}{2}}\{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2\}^{\frac{1}{2}}.$$

**2.677** The direction cosines of the binormal to the curve in the form (b) are:

$$l'' = \frac{y'z'' - z'y''}{S},$$

$$m'' = \frac{z'x'' - x'z''}{S},$$

$$n'' = \frac{x'y'' - y'x''}{S},$$

where

$$S = \{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2\}^{\frac{1}{2}}.$$

**2.678** If $s$, the distance measured along the curve from a fixed point on it is the parameter, $t$:

$$l' = \rho\frac{d^2x}{ds^2},\quad m' = \rho\frac{d^2y}{ds^2},\quad n' = \rho\frac{d^2z}{ds^2},$$

where $\rho$ is the principal radius of curvature; and

$$l'' = \rho\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right),$$

$$m'' = \rho\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right),$$

$$n'' = \rho\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right).$$

**2.679** The radius of torsion, or radius of second curvature of a space curve is:

$$\tau = \frac{(x'^2 + y'^2 + z'^2)^{\frac{1}{2}}}{\left\{\left(\frac{\partial l''}{\partial t}\right)^2 + \left(\frac{\partial m''}{\partial t}\right)^2 + \left(\frac{\partial n''}{\partial t}\right)^2\right\}^{\frac{1}{2}}}$$

$$= -\frac{1}{S^2}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

where $S$ is given in **2.677**.

**2.680** When $t = s$:

$$\frac{1}{\tau} = \left\{\left(\frac{\partial l''}{\partial s}\right)^2 + \left(\frac{\partial m''}{\partial s}\right)^2 + \left(\frac{\partial n''}{\partial s}\right)^2\right\}^{\frac{1}{2}}$$

$$= -\rho^2 \begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\ \frac{d^3x}{ds^3} & \frac{d^3y}{ds^3} & \frac{d^3z}{ds^3} \end{vmatrix}.$$

**2.681** The direction cosines of the tangent to a space curve in the form (c) are:

$$l,\ m,\ n = \frac{1,\ y',\ z'}{\sqrt{1 + y'^2 + z'^2}}$$

where accents denote differentials with respect to $x$:

$$y' = \frac{d\phi(x)}{dx}, \quad z' = \frac{d\psi(x)}{dx}.$$

**2.682** The principal radius of curvature of a space curve in the form (c) is:

$$\rho = \left\{ \frac{(1 + y'^2 + z'^2)^3}{(y'z'' - z'y'')^2 + y''^2 + z''^2} \right\}^{\frac{1}{2}}.$$

**2.683** The radius of torsion of a space curve in the form (c) is:

$$\tau = \frac{(1 + y'^2 + z'^2)^3}{\rho^2(y''z''' - z''y''')}.$$

**2.690** The relation between the direction cosines of the tangent, principal normal and binormal to a space curve is:

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} = 1.$$

**2.691** The tangent, principal normal and binormal all being mutually perpendicular the relations of **2.00** hold among their direction cosines.

# III. TRIGONOMETRY

**3.00** $\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$, $\csc x = \dfrac{1}{\sin x}$, $\cot x = \dfrac{1}{\tan x}$,

$\sec^2 x = 1 + \tan^2 x$, $\csc^2 x = 1 + \cot^2 x$, $\sin^2 x + \cos^2 x = 1$,

$\text{versin } x = 1 - \cos x$, $\text{coversin } x = 1 - \sin x$, $\text{haversin } x = \sin^2 \dfrac{x}{2}$.

**3.01** $\sin x = -\sin(-x) = \sqrt{\dfrac{1 - \cos 2x}{2}}, = 2\sqrt{\cos^2 \dfrac{x}{2} - \cos^4 \dfrac{x}{2}},$

$$= 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}},$$

$$= \frac{1}{\sqrt{1 + \cot^2 x}} = \frac{1}{\cot \frac{x}{2} - \cot x} = \frac{1}{\tan \frac{x}{2} + \cot x},$$

$$= \cot \frac{x}{2} \cdot (1 - \cos x) = \tan \frac{x}{2} \cdot (1 + \cos x),$$

$$= \sin y \cos (x - y) + \cos y \sin (x - y),$$

$$= \cos y \sin (x + y) - \sin y \cos (x + y),$$

$$= -\tfrac{1}{2} i \, (e^{ix} - e^{-ix}).$$

**3.02** $\cos x = \cos(-x) = \sqrt{\dfrac{1 + \cos 2x}{2}} = 1 - 2 \sin^2 \dfrac{x}{2},$

$$= \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = 2 \cos^2 \frac{x}{2} - 1 = \frac{1}{\sqrt{1 + \tan^2 x}},$$

$$= \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1}{1 + \tan x \tan \frac{x}{2}} = \frac{1}{\tan x \cot \frac{x}{2} - 1},$$

$$= \frac{\cot \frac{x}{2} - \tan \frac{x}{2}}{\cot \frac{x}{2} + \tan \frac{x}{2}} = \frac{\cot x}{\sqrt{1 + \cot^2 x}} = \frac{\sin 2x}{2 \sin x},$$

$$= \cos y \cos (x + y) + \sin y \sin (x + y),$$

$$= \cos y \cos (x - y) - \sin y \sin (x - y),$$

$$= \tfrac{1}{2}(e^{ix} + e^{-ix}).$$

**3.03** $$\tan x = -\tan(-x) = \frac{\sin 2x}{1+\cos 2x} = \frac{1-\cos 2x}{\sin 2x}, =$$

$$\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \frac{\sin(x+y)+\sin(x-y)}{\cos(x+y)+\cos(x-y)},$$

$$= \frac{\cos(x-y)-\cos(x+y)}{\sin(x+y)-\sin(x-y)} = \cot x - 2\cot 2x,$$

$$= \frac{\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + \frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}},$$

$$= \frac{1}{1-\tan\frac{x}{2}} - \frac{1}{1+\tan\frac{x}{2}},$$

$$= i\,\frac{1-e^{2ix}}{1+e^{2ix}}.$$

**3.04** The values of five trigonometric functions in terms of the sixth are given in the following table. (For signs, see **3.05.**)

| | $\sin x = a$ | $\cos x = a$ | $\tan x = a$ | $\cot x = a$ | $\sec x = a$ | $\csc x = a$ |
|---|---|---|---|---|---|---|
| $\sin x =$ | $a$ | $\sqrt{1-a^2}$ | $\frac{a}{\sqrt{1+a^2}}$ | $\frac{1}{\sqrt{1+a^2}}$ | $\frac{\sqrt{a^2-1}}{a}$ | $\frac{1}{a}$ |
| $\cos x =$ | $\sqrt{1-a^2}$ | $a$ | $\frac{1}{\sqrt{1+a^2}}$ | $\frac{a}{\sqrt{1+a^2}}$ | $\frac{1}{a}$ | $\frac{\sqrt{a^2-1}}{a}$ |
| $\tan x =$ | $\frac{a}{\sqrt{1-a^2}}$ | $\frac{\sqrt{1-a^2}}{a}$ | $a$ | $\frac{1}{a}$ | $\sqrt{a^2-1}$ | $\frac{1}{\sqrt{a^2-1}}$ |
| $\cot x =$ | $\frac{\sqrt{1-a^2}}{a}$ | $\frac{a}{\sqrt{1-a^2}}$ | $\frac{1}{a}$ | $a$ | $\frac{1}{\sqrt{a^2-1}}$ | $\sqrt{a^2-1}$ |
| $\sec x =$ | $\frac{1}{\sqrt{1-a^2}}$ | $\frac{1}{a}$ | $\sqrt{1+a^2}$ | $\frac{\sqrt{1+a^2}}{a}$ | $a$ | $\frac{a}{\sqrt{a^2-1}}$ |
| $\csc x =$ | $\frac{1}{a}$ | $\frac{1}{\sqrt{1-a^2}}$ | $\frac{\sqrt{1+a^2}}{a}$ | $\sqrt{1+a^2}$ | $\frac{a}{\sqrt{a^2-1}}$ | $a$ |

**3.05** The trigonometric functions are periodic, the periods of the sin, cos, sec, csc being $2\pi$, and those of the tan and cot, $\pi$. Their signs may be determined from the following table. In using formulas giving any of the trigonometric

functions by the root of some quantity, the proper sign may be taken from this table.

| | $0°$ | $0 - \frac{\pi}{2}$<br>$0 - 90°$ | $\frac{\pi}{2}$<br>$90°$ | $\frac{\pi}{2} - \pi$<br>$90° - 180°$ | $\pi$<br>$180°$ | $\pi - \frac{3}{2}\pi$<br>$180° - 270°$ | $\frac{3}{2}\pi$<br>$270°$ | $\frac{3}{2}\pi - 2\pi$<br>$270° - 360°$ | $2\pi$<br>$360°$ |
|---|---|---|---|---|---|---|---|---|---|
| sin | 0 | + | 1 | + | 0 | − | −1 | − | 0 |
| cos | 1 | + | 0 | − | −1 | − | 0 | + | 1 |
| tan | 0 | + | $\pm\infty$ | − | 0 | + | $\pm\infty$ | − | 0 |
| cot | $\mp\infty$ | + | 0 | − | $\mp\infty$ | + | 0 | − | $\mp\infty$ |
| sec | 1 | + | $\pm\infty$ | − | −1 | − | $\pm\infty$ | + | 1 |
| csc | $\mp\infty$ | + | 1 | + | $\pm\infty$ | − | −1 | − | $\mp\infty$ |

**3.10** Functions of Half an Angle. (See **3.05** for signs.)

**3.101**
$$\sin \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{2}},$$
$$= \frac{1}{2}\left\{ \pm\sqrt{1 + \sin x} \mp \sqrt{1 - \sin x} \right\}$$
$$= \pm\sqrt{\frac{1}{2}\left(1 - \frac{1}{\pm\sqrt{1 + \tan^2 x}}\right)}$$

**3.102**
$$\cos \tfrac{1}{2}x = \pm\sqrt{\frac{1 + \cos x}{2}},$$
$$= \frac{1}{2}\left\{ \pm\sqrt{1 + \sin x} \pm \sqrt{1 - \sin x} \right\},$$
$$= \pm\sqrt{\frac{1}{2}\left(1 + \frac{1}{\pm\sqrt{1 + \tan^2 x}}\right)}.$$

**3.103**
$$\tan \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

$$= \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x},$$

$$= \frac{\pm\sqrt{1 + \tan^2 x} - 1}{\tan x}.$$

**3.11** Functions of the Sum and Difference of Two Angles.

**3.111**
$$\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$= \cos x \cos y (\tan x \pm \tan y),$$
$$= \frac{\tan x \pm \tan y}{\tan x \mp \tan y} \sin (x \mp y),$$
$$= \frac{1}{2}\left\{\cos (x + y) + \cos (x - y)\right\} (\tan x \pm \tan y).$$

**3.112**
$$\cos (x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$= \cos x \cos y (1 \mp \tan x \tan y),$$
$$= \frac{\cot x \mp \tan y}{\cot x \pm \tan y} \cos (x \mp y),$$
$$= \frac{\cot y \mp \tan x}{\cot y \tan x \mp 1} \sin (x \mp y),$$
$$= \cos x \sin y (\cot y \mp \tan x).$$

**3.113**
$$\tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$= \frac{\cot y \pm \cot x}{\cot x \cot y \mp 1},$$
$$= \frac{\sin 2x \pm \sin 2y}{\cos 2x + \cos 2y}.$$

**3.114**
$$\cot (x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x},$$
$$= -\frac{\sin 2x \mp \sin 2y}{\cos 2x - \cos 2y}.$$

**3.115** The cosine and sine of the sum of any number of angles in terms of the sine and cosine of the angles are given by the real and imaginary parts of

$$\cos (x_1 + x_2 + \ldots + x_n) + i \sin (x_1 + x_2 + \ldots + x_n)$$
$$= (\cos x_1 + i \sin x_1)(\cos x_2 + i \sin x_2) \ldots\ldots (\cos x_n + i \sin x_n)$$

## 3.12 Sums and Differences of Trigonometric Functions.

**3.121**
$$\begin{aligned}\sin x \pm \sin y &= 2 \sin \tfrac{1}{2}(x \pm y) \cos \tfrac{1}{2}(x \mp y),\\ &= (\cos x + \cos y) \tan \tfrac{1}{2}(x \pm y),\\ &= (\cos y - \cos x) \cot \tfrac{1}{2}(x \mp y),\\ &= \frac{\tan \tfrac{1}{2}(x \pm y)}{\tan \tfrac{1}{2}(x \mp y)} (\sin x \mp \sin y).\end{aligned}$$

**3.122**
$$\begin{aligned}\cos x + \cos y &= 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y),\\ &= \frac{\sin x \pm \sin y}{\tan \tfrac{1}{2}(x \pm y)},\\ &= \frac{\cot \tfrac{1}{2}(x + y)}{\tan \tfrac{1}{2}(x - y)} (\cos y - \cos x).\end{aligned}$$

**3.123**
$$\begin{aligned}\cos x - \cos y &= 2 \sin \tfrac{1}{2}(y + x) \sin \tfrac{1}{2}(y - x)\\ &= -(\sin x \pm \sin y) \tan \tfrac{1}{2}(x \mp y).\end{aligned}$$

**3.124**
$$\begin{aligned}\tan x \pm \tan y &= \frac{\sin (x \pm y)}{\cos x \cdot \cos y}.\\ &= \frac{\sin (x \pm y)}{\sin (x \mp y)} (\tan x \mp \tan y),\\ &= \tan y \tan (x \pm y)(\cot y \mp \tan x),\\ &= \frac{1 \mp \tan x \tan y}{\cot (x \pm y)},\\ &= (1 \mp \tan x \tan y) \tan (x \pm y).\end{aligned}$$

**3.125**
$$\cot x \pm \cot y = \pm \frac{\sin (x \pm y)}{\sin x \sin y}.$$

**3.130**

1. $$\frac{\sin x \pm \sin y}{\cos x + \cos y} = \tan \tfrac{1}{2}(x \pm y).$$

2. $$\frac{\sin x \pm \sin y}{\cos x - \cos y} = -\cot \tfrac{1}{2}(x \mp y).$$

3. $$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\tan \tfrac{1}{2}(x + y)}{\tan \tfrac{1}{2}(x - y)}.$$

**3.140**

1. $\sin^2 x + \sin^2 y = 1 - \cos(x+y)\cos(x-y).$
2. $\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y)\sin(x-y).$
3. $\cos^2 x - \sin^2 y = \cos(x+y)\cos(x-y).$
4. $\sin^2(x+y) + \sin^2(x-y) = 1 - \cos 2x \cos 2y.$
5. $\sin^2(x+y) - \sin^2(x-y) = \sin 2x \sin 2y.$
6. $\cos^2(x+y) + \cos^2(x-y) = 1 + \cos 2x \cos 2y.$
7. $\cos^2(x+y) - \cos^2(x-y) = -\sin 2x \sin 2y.$

**3.150**

1. $\cos nx \cos mx = \frac{1}{2}\cos(n-m)x + \frac{1}{2}\cos(n+m)x.$
2. $\sin nx \sin mx = \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x.$
3. $\cos nx \sin mx = \frac{1}{2}\sin(n+m)x - \frac{1}{2}\sin(n-m)x.$

**3.160**

1. $e^{x+iy} = e^x(\cos y + i \sin y).$
2. $a^{x+iy} = a^x\{\cos(y \log a) + i \sin(y \log a)\}.$
3. $(\cos x \pm i \sin x)^n = \cos nx \pm i \sin nx$ [De Moivre's Theorem].
4. $\sin(x \pm iy) = \sin x \cosh y \pm i \cos x \sinh y.$
5. $\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y.$
6. $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}).$
7. $\sin x = -\frac{i}{2}(e^{ix} - e^{-ix}).$
8. $e^{ix} = \cos x + i \sin x.$
9. $e^{-ix} = \cos x - i \sin x.$

**3.170** Sines and Cosines of Multiple Angles.

**3.171** $n$ an even integer:

$$\sin nx = n \cos x \left\{ \sin x - \frac{(n^2-2^2)}{3!}\sin^3 x + \frac{(n^2-2^2)(n^2-4^2)}{5!}\sin^5 x - \ldots \right\}.$$

$$\cos nx = 1 - \frac{n^2}{2!}\sin^2 x + \frac{n^2(n^2-2^2)}{4!}\sin^4 x - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}\sin^6 x + \ldots$$

**3.172** $n$ an odd integer:

$$\sin nx = n\left\{\sin x - \frac{(n^2-1^2)}{3!}\sin^3 x + \frac{(n^2-1^2)(n^2-3^2)}{5!}\sin^5 x - \ldots\right\}.$$

$$\cos nx = \cos x\left\{1 - \frac{(n^2-1^2)}{2!}\sin^2 x + \frac{(n^2-1^2)(n^2-3^2)}{4!}\sin^4 x - \ldots\right\}.$$

**3.173** $n$ an even integer:

$$\sin nx = (-1)^{\frac{n}{2}-1}\cos x\left\{2^{n-1}\sin^{n-1}x - \frac{(n-2)}{1!}2^{n-3}\sin^{n-3}x + \frac{(n-3)(n-4)}{2!}2^{n-5}\sin^{n-5}x - \frac{(n-4)(n-5)(n-6)}{3!}2^{n-7}\sin^{n-7}x + \ldots\right\}.$$

$$\cos nx = (-1)^{\frac{n}{2}}\left\{2^{n-1}\sin^n x - \frac{n}{1!}2^{n-3}\sin^{n-2}x + \frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}x - \frac{n(n-3)(n-5)}{3!}2^{n-7}\sin^{n-6}x + \ldots\right\}.$$

**3.174** $n$ an odd integer:

$$\sin nx = (-1)^{\frac{n-1}{2}}\left\{2^{n-1}\sin^n x - \frac{n}{1!}2^{n-3}\sin^{n-2}x + \frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}x - \frac{n(n-3)(n-5)}{3!}2^{n-7}\sin^{n-6}x + \ldots\right\}.$$

$$\cos nx = (-1)^{\frac{n-1}{2}}\cos x\left\{2^{n-1}\sin^{n-1}x - \frac{n-2}{1!}2^{n-3}\sin^{n-3}x + \frac{(n-3)(n-4)}{2!}2^{n-5}\sin^{n-5}x - \frac{(n-4)(n-5)(n-6)}{3!}2^{n-7}\sin^{n-7}x + \ldots\right\}.$$

**3.175** $n$ any integer:

$$\sin nx = \sin x\left\{2^{n-1}\cos^{n-1}x - \frac{n-2}{1!}2^{n-3}\cos^{n-3}x + \frac{(n-3)(n-4)}{2!}2^{n-5}\cos^{n-5}x - \frac{(n-4)(n-5)(n-6)}{3!}2^{n-7}\cos^{n-7}x + \ldots\right\}.$$

$$\cos nx = 2^{n-1}\cos^n x - \frac{n}{1!}2^{n-3}\cos^{n-2}x + \frac{n(n-3)}{2!}2^{n-5}\cos^{n-4}x - \frac{n(n-4)(n-5)}{3!}2^{n-7}\cos^{n-6}x + \ldots.$$

**3.176**

$$\sin 2x = 2 \sin x \cos x.$$
$$\sin 3x = \sin x(3 - 4 \sin^2 x)$$
$$= \sin x(4 \cos^2 x - 1).$$
$$\sin 4x = \sin x(8 \cos^3 x - 4 \cos x).$$
$$\sin 5x = \sin x(5 - 20 \sin^2 x + 16 \sin^4 x)$$
$$= \sin x(16 \cos^4 x - 12 \cos^2 x + 1).$$
$$\sin 6x = \sin x(32 \cos^5 x - 32 \cos^3 x + 6 \cos x).$$

**3.177**

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2 \sin^2 x$$
$$= 2 \cos^2 x - 1.$$
$$\cos 3x = \cos x(4 \cos^2 x - 3)$$
$$= \cos x(1 - 4 \sin^2 x).$$
$$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1.$$
$$\cos 5x = \cos x(16 \cos^4 x - 20 \cos^2 x + 5)$$
$$= \cos x(16 \sin^4 x - 12 \sin^2 x + 1).$$
$$\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1.$$

**3.178**

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$
$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.$$

**3.180** Integral Powers of Sine and Cosine.

**3.181** $n$ an even integer:

$$\sin^n x = \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} \left\{ \cos nx - n \cos (n-2)x + \frac{n(n-1)}{2!} \cos (n-4)x \right.$$
$$\left. - \frac{n(n-1)(n-2)}{3!} \cos (n-6)x + \cdots\cdots + (-1)^{\frac{n}{2}} \frac{1}{2} \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!} \right\}$$

$$\cos^n x = \frac{1}{2^{n-1}} \left\{ \cos nx + n \cos (n-2)x + \frac{n(n-1)}{2!} \cos (n-4)x \right.$$
$$\left. + \frac{n(n-1)(n-2)}{3!} \cos (n-6)x + \cdots + \frac{1}{2} \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!} \cdot \right\}$$

**3.182** $n$ an odd integer:

$$\sin^n x = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}}\left\{\sin nx - n\sin(n-2)x + \frac{n(n-1)}{2!}\sin(n-4)x\right.$$

$$\left. - \frac{n(n-1)(n-2)}{3!}\sin(n-6)x + \cdots + (-1)^{\frac{n-1}{2}}\frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\sin x\right\}.$$

$$\cos^n x = \frac{1}{2^{n-1}}\left\{\cos nx + n\cos(n-2)x + \frac{n(n-1)}{2!}\cos(n-4)x\right.$$

$$\left. + \frac{n(n-1)(n-2)}{3!}\cos(n-6)x + \cdots + \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\cos x\right\}.$$

**3.183**

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$
$$\sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x).$$
$$\sin^4 x = \tfrac{1}{8}(\cos 4x - 4\cos 2x + 3).$$
$$\sin^5 x = \tfrac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x).$$
$$\sin^6 x = -\tfrac{1}{32}(\cos 6x - 6\cos 4x + 15\cos 2x - 10).$$

**3.184**

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x).$$
$$\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x).$$
$$\cos^4 x = \tfrac{1}{8}(3 + 4\cos 2x + \cos 4x).$$
$$\cos^5 x = \tfrac{1}{16}(10\cos x + 5\cos 3x + \cos 5x).$$
$$\cos^6 x = \tfrac{1}{32}(10 + 15\cos 2x + 6\cos 4x + \cos 6x).$$

## INVERSE CIRCULAR FUNCTIONS

**3.20** The inverse circular and logarithmic functions are multiple valued; i.e., if

$$0 < \sin^{-1} x < \frac{\pi}{2},$$

the solution of $x = \sin\theta$ is:

$$\theta = 2n\pi + \sin^{-1} x,$$

where $n$ is a positive integer. In the following formulas the cyclic constants are omitted.

**3.21**

$$\sin^{-1} x = -\sin^{-1}(-x) = \frac{\pi}{2} - \cos^{-1} x = \cos^{-1}\sqrt{1-x^2}$$

$$= \frac{\pi}{2} - \sin^{-1}\sqrt{1-x^2} = \frac{\pi}{4} + \frac{1}{2}\sin^{-1}(2x^2-1)$$

$$= \frac{1}{2}\cos^{-1}(1-2x^2) = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= 2\tan^{-1}\left\{\frac{1-\sqrt{1-x^2}}{x}\right\} = \frac{1}{2}\tan^{-1}\left\{\frac{2x\sqrt{1-x^2}}{1-2x^2}\right\}$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \frac{\pi}{2} - i\log(x+\sqrt{x^2-1}).$$

**3.22**

$$\cos^{-1} x = \pi - \cos^{-1}(-x) = \frac{\pi}{2} - \sin^{-1} x = \frac{1}{2}\cos^{-1}(2x^2-1)$$

$$= 2\cos^{-1}\sqrt{\frac{1+x}{2}} = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$= 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \frac{1}{2}\tan^{-1}\left\{\frac{2x\sqrt{1-x^2}}{2x^2-1}\right\} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= i\log(x+\sqrt{x^2-1}) = \pi - i\log(\sqrt{x^2-1}-x).$$

**3.23**

$$\tan^{-1} x = -\tan^{-1}(-x) = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} = \frac{\pi}{2} - \cot^{-1} x = \sec^{-1}\sqrt{1+x^2}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{1}{x} = \frac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}$$

$$= 2\cos^{-1}\left\{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}} = 2\sin^{-1}\left\{\frac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$$

$$= -\tan^{-1} c + \tan^{-1}\frac{x+c}{1-cx}$$

$$= \frac{1}{2}i\log\frac{1-ix}{1+ix} = \frac{1}{2}i\log\frac{i+x}{i-x} = -\frac{1}{2}i\log\frac{1+ix}{1-ix}.$$

**3.25**

1. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\{x\sqrt{1 - y^2} \pm y\sqrt{1 - x^2}\}.$

2. $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\{xy \mp \sqrt{(1 - x^2)\,(1 - y^2)}\}.$

3. $\sin^{-1} x \pm \cos^{-1} y = \sin^{-1}\{xy \pm \sqrt{(1 - x^2)\,(1 - y^2)}\}$
$= \cos^{-1}\{y\sqrt{1 - x^2} \mp x\sqrt{1 - y^2}\}.$

4. $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \dfrac{x \pm y}{1 \mp xy}.$

5. $\tan^{-1} x \pm \cot^{-1} y = \tan^{-1} \dfrac{xy \pm 1}{y \mp x}$
$= \cot^{-1} \dfrac{y \mp x}{xy \pm 1}.$

## HYPERBOLIC FUNCTIONS

**3.30** Formulas for the hyperbolic functions may be obtained from the corresponding formulas for the circular functions by replacing $x$ by $ix$ and using the following relations:

1. $\sin ix = \frac{1}{2}i(e^x - e^{-x}) = i \sinh x.$

2. $\cos ix = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$

3. $\tan ix = \dfrac{i(e^{2x} - 1)}{e^{2x} + 1} = i \tanh x.$

4. $\cot ix = -i\,\dfrac{e^{2x} + 1}{e^{2x} - 1} = -i \coth x.$

5. $\sec ix = \dfrac{2}{e^x + e^{-x}} = \operatorname{sech} x.$

6. $\csc ix = -\dfrac{2i}{e^x - e^{-x}} = -i \operatorname{csch} x.$

7. $\sin^{-1} ix = i \sinh^{-1} x = i \log (x + \sqrt{1 + x^2}).$

8. $\cos^{-1} ix = -i \cosh^{-1} x = \dfrac{\pi}{2} - i \log (x + \sqrt{1 + x^2}).$

9. $\tan^{-1} ix = i \tanh^{-1} x = i \log \sqrt{\dfrac{1 + x}{1 - x}}.$

10. $\cot^{-1} ix = -i \coth^{-1} x = -i \log \sqrt{\dfrac{x + 1}{x - 1}}.$

**3.310** The values of five hyperbolic functions in terms of the sixth are given in the following table:

| | $\sinh x = a$ | $\cosh x = a$ | $\tanh x = a$ | $\coth x = a$ | $\operatorname{sech} x = a$ | $\operatorname{csch} x = a$ |
|---|---|---|---|---|---|---|
| $\sinh x =$ | $a$ | $\sqrt{a^2 - 1}$ | $\frac{a}{\sqrt{1 - a^2}}$ | $\frac{1}{\sqrt{a^2 - 1}}$ | $\frac{\sqrt{1 - a^2}}{a}$ | $\frac{1}{a}$ |
| $\cosh x =$ | $\sqrt{1 + a^2}$ | $a$ | $\frac{1}{\sqrt{1 - a^2}}$ | $\frac{a}{\sqrt{a^2 - 1}}$ | $\frac{1}{a}$ | $\frac{\sqrt{1 + a^2}}{a}$ |
| $\tanh x =$ | $\frac{a}{\sqrt{1 + a^2}}$ | $\frac{\sqrt{a^2 - 1}}{a}$ | $a$ | $\frac{1}{a}$ | $\sqrt{1 - a^2}$ | $\frac{1}{\sqrt{1 + a^2}}$ |
| $\coth x =$ | $\frac{\sqrt{a^2 + 1}}{a}$ | $\frac{a}{\sqrt{a^2 - 1}}$ | $\frac{1}{a}$ | $a$ | $\frac{1}{\sqrt{1 - a^2}}$ | $\sqrt{1 + a^2}$ |
| $\operatorname{sech} x =$ | $\frac{1}{\sqrt{1 + a^2}}$ | $\frac{1}{a}$ | $\sqrt{1 - a^2}$ | $\frac{\sqrt{a^2 - 1}}{a}$ | $a$ | $\frac{a}{\sqrt{1 + a^2}}$ |
| $\operatorname{csch} x =$ | $\frac{1}{a}$ | $\frac{1}{\sqrt{a^2 - 1}}$ | $\frac{\sqrt{1 - a^2}}{a}$ | $\sqrt{a^2 - 1}$ | $\frac{a}{\sqrt{1 - a^2}}$ | $a$ |

**3.311** Periodicity of the Hyperbolic Functions.

The functions $\sinh x$, $\cosh x$, $\operatorname{sech} x$, $\operatorname{csch} x$ have an imaginary period $2\pi i$, e.g.:

$$\cosh x = \cosh (x + 2\pi i n),$$

where $n$ is any integer. The functions $\tanh x$, $\coth x$ have an imaginary period $\pi i$.

The values of the hyperbolic functions for the argument $o$, $\frac{\pi}{2} i$, $\pi i$, $\frac{3\pi i}{2}$, are given in the following table:

| | $0$ | $\frac{\pi}{2} i$ | $\pi i$ | $3\frac{\pi}{2} i$ |
|---|---|---|---|---|
| sinh | $0$ | $i$ | $0$ | $-i$ |
| cosh | $1$ | $0$ | $-1$ | $0$ |
| tanh | $0$ | $\infty \cdot i$ | $0$ | $\infty \cdot i$ |
| coth | $\infty$ | $0$ | $\infty$ | $0$ |
| sech | $1$ | $\infty$ | $-1$ | $\infty$ |
| csch | $\infty$ | $-i$ | $\infty$ | $i$ |

**3.320**

1. $$\sinh \tfrac{1}{2}x = \sqrt{\frac{\cosh x - 1}{2}}$$

2. $$\cosh \tfrac{1}{2}x = \sqrt{\frac{\cosh x + 1}{2}}$$

3. $$\tanh \tfrac{1}{2}x = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}}.$$

**3.33**

1. $$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$$

2. $$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

3. $$\tanh (x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

4. $$\coth (x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}.$$

**3.34**

1. $$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y).$$

2. $$\sinh x - \sinh y = 2 \cosh \tfrac{1}{2}(x + y) \sinh \tfrac{1}{2}(x - y).$$

3. $$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y).$$

4. $$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x + y) \sinh \tfrac{1}{2}(x - y).$$

5. $$\tanh x + \tanh y = \frac{\sinh (x + y)}{\cosh x \cosh y}.$$

6. $$\tanh x - \tanh y = \frac{\sinh (x - y)}{\cosh x \cosh y}.$$

7. $$\coth x + \coth y = \frac{\sinh (x + y)}{\sinh x \sinh y}.$$

8. $$\coth x - \coth y = -\frac{\sinh (x - y)}{\sinh x \sinh y}.$$

### 3.35

1. $\sinh(x+y) + \sinh(x-y) = 2\sinh x \cosh y.$
2. $\sinh(x+y) - \sinh(x-y) = 2\cosh x \sinh y.$
3. $\cosh(x+y) + \cosh(x-y) = 2\cosh x \cosh y.$
4. $\cosh(x+y) - \cosh(x-y) = 2\sinh x \sinh y.$
5. $\tanh \tfrac{1}{2}(x \pm y) = \dfrac{\sinh x \pm \sinh y}{\cosh x + \cosh y}.$
6. $\coth \tfrac{1}{2}(x \pm y) = \dfrac{\sinh x \mp \sinh y}{\cosh x - \cosh y}.$
7. $\dfrac{\tanh x + \tanh y}{\tanh x - \tanh y} = \dfrac{\sinh(x+y).}{\sinh(x-y).}$
8. $\dfrac{\coth x + \coth y}{\coth x - \coth y} = -\dfrac{\sinh(x+y)}{\sinh(x-y)}.$

### 3.36

1. $\sinh(x+y) + \cosh(x+y) = (\cosh x + \sinh x)(\cosh y + \sinh y).$
2. $\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$
   $= \cosh^2 x - \cosh^2 y.$
3. $\cosh(x+y)\cosh(x-y) = \cosh^2 x + \sinh^2 y$
   $= \sinh^2 x + \cosh^2 y.$
4. $\sinh x + \cosh x = \dfrac{1 + \tanh \frac{1}{2}x}{1 - \tanh \frac{1}{2}x}.$
5. $(\sinh x + \cosh x)^n = \cosh nx + \sinh nx.$

### 3.37

1. $e^x = \cosh x + \sinh x.$
2. $e^{-x} = \cosh x - \sinh x.$
3. $\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$
4. $\cosh x = \tfrac{1}{2}(e^x + e^{-x}).$

**3.38**

1. $$\sinh 2x = 2 \sinh x \cosh x,$$
$$= \frac{2 \tanh x}{1 - \tanh^2 x}.$$

2. $$\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1,$$
$$= 1 + 2 \sinh^2 x,$$
$$= \frac{1 + \tanh^2 x}{1 - \tanh^2 x}.$$

3. $$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

4. $$\sinh 3x = 3 \sinh x + 4 \sinh^3 x.$$

5. $$\cosh 3x = 4 \cosh^3 x - 3 \cosh x.$$

6. $$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}.$$

**3.40** Inverse Hyperbolic Functions.

The hyperbolic functions being periodic, the inverse functions are multiple valued (**3.311**). In the following formulas the periodic constants are omitted, the principal values only being given.

1. $$\sinh^{-1} x = \log (x + \sqrt{x^2 + 1}) = \cosh^{-1} \sqrt{x^2 + 1}.$$

2. $$\cosh^{-1} x = \log (x + \sqrt{x^2 - 1}) = \sinh^{-1} \sqrt{x^2 - 1}.$$

3. $$\tanh^{-1} x = \log \sqrt{\frac{1 + x}{1 - x}}.$$

4. $$\coth^{-1} x = \log \sqrt{\frac{x + 1}{x - 1}} = \tanh^{-1} \frac{1}{x}.$$

5. $$\operatorname{sech}^{-1} x = \log \left( \frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right) = \cosh^{-1} \frac{1}{x}.$$

6. $$\operatorname{csch}^{-1} x = \log \left( \frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) = \sinh^{-1} \frac{1}{x}.$$

**3.41**

1. $$\sinh^{-1} x \pm \sinh^{-1} y = \sinh^{-1}(x\sqrt{1 + y^2} \pm y\sqrt{1 + x^2}).$$

2. $$\cosh^{-1} x \pm \cosh^{-1} y = \cosh^{-1} (xy \pm \sqrt{(x^2 - 1)(y^2 - 1)}).$$

3. $$\tanh^{-1} x \pm \tanh^{-1} y = \tanh^{-1} \frac{x \pm y}{1 \pm xy}.$$

**3.42**

1. $$\cosh^{-1}\tfrac{1}{2}\left(x+\frac{1}{x}\right)=\sinh^{-1}\tfrac{1}{2}\left(x-\frac{1}{x}\right),$$
$$=\tanh^{-1}\frac{x^2-1}{x^2+1}=2\tanh^{-1}\frac{x-1}{x+1},$$
$$=\log x.$$

2. $$\cosh^{-1}\csc 2x=-\sinh^{-1}\cot 2x=-\tanh^{-1}\cos 2x,$$
$$=\log\tan x.$$

3. $$\tanh^{-1}\tan^2\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{1}{1}\log\csc x.$$

4. $$\tanh^{-1}\tan^2\frac{x}{2}=\frac{1}{2}\log\sec x.$$

**3.43** The Gudermannian.

If,

1. $$\cosh x=\sec\theta.$$

2. $$\sinh x=\tan\theta.$$

3. $$e^x=\sec\theta+\tan\theta=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

4. $$x=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

5. $$\theta=\operatorname{gd}x.$$

**3.44**

1. $$\sinh x=\tan\operatorname{gd}x.$$

2. $$\cosh x=\sec\operatorname{gd}x.$$

3. $$\tanh x=\sin\operatorname{gd}x.$$

4. $$\tanh\frac{x}{2}=\tan\frac{1}{2}\operatorname{gd}x.$$

5. $$e^x=\frac{1+\sin\operatorname{gd}x}{\cos\operatorname{gd}x}=\frac{1-\cos\left(\frac{\pi}{2}+\operatorname{gd}x\right)}{\sin\left(\frac{\pi}{2}+\operatorname{gd}x\right)}.$$

6. $\tanh^{-1} \tan x = \frac{1}{2} \operatorname{gd} 2x.$

7. $\tan^{-1} \tanh x = \frac{1}{2} \operatorname{gd}^{-1} 2x.$

## 3.50 SOLUTION OF OBLIQUE PLANE TRIANGLES

$a, b, c$ = Sides of triangle,

$\alpha, \beta, \gamma$ = angles opposite to $a, b, c$, respectively,

$A$ = area of triangle,

$s = \frac{1}{2}(a + b + c)$.

| *Given* | *Sought* | *Formula* |
|---|---|---|
| $a, b, c$ | $\alpha$ | $\sin \frac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}}.$ |
| | | $\cos \frac{1}{2}\alpha = \sqrt{\frac{s(s-a)}{bc}}.$ |
| | | $\tan \frac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$ |
| | | $\cos \alpha = \frac{c^2 + b^2 - a^2}{2bc}.$ |
| | $A$ | $A = \sqrt{s(s-a)(s-b)(s-c)}.$ |
| $a, b, \alpha$ | $\beta$ | $\sin \beta = \frac{b \sin \alpha}{a}.$ |

When $a > b$, $\beta < \frac{\pi}{2}$ and but one value results. When $b > a$ $\beta$ has two values.

| | | |
|---|---|---|
| | $\gamma$ | $\gamma = 180° - (\alpha + \beta).$ |
| | $c$ | $c = \frac{a \sin \gamma}{\sin \alpha}.$ |
| | $A$ | $A = \frac{1}{2} ab \sin \gamma.$ |
| $a, \alpha, \beta$ | $b$ | $b = \frac{a \sin \beta}{\sin \alpha}$ |
| | $\gamma$ | $\gamma = 180° - (\alpha + \beta).$ |
| | $c$ | $c = \frac{a \sin \gamma}{\sin \alpha} = \frac{a \sin (\alpha + \beta)}{\sin \alpha}.$ |

| *Given* | *Sought* | *Formula* |
|---|---|---|
| | $A$ | $A = \frac{1}{2} ab \sin \gamma = \frac{1}{2} a^2 \frac{\sin \beta \sin \gamma}{\sin \alpha}.$ |
| $a, b, \gamma$ | $\alpha$ | $\tan \alpha = \frac{a \sin \gamma}{b - a \cos \gamma}.$ |
| | $\alpha, \beta$ | $\frac{1}{2}(\alpha + \beta) = 90^\circ - \frac{1}{2}\gamma.$ |
| | | $\tan \frac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \cot \frac{1}{2}\gamma$ |
| | $c$ | $c = (a^2 + b^2 - 2ab \cos \gamma)^{\frac{1}{2}}.$ |
| | | $= \{(a + b)^2 - 4ab \cos^2 \frac{1}{2}\gamma\}^{\frac{1}{2}}$ |
| | | $= \{(a - b)^2 + 4ab \sin^2 \frac{1}{2}\gamma\}^{\frac{1}{2}}.$ |
| | | $= \frac{a - b}{\cos \phi}$ where $\tan \phi = 2\sqrt{ab} \frac{\sin \frac{1}{2}\gamma}{a - b}$ |
| | | $= \frac{a \sin \gamma}{\sin \alpha}.$ |
| | $A$ | $A = \frac{1}{2} ab \sin \gamma.$ |

## SOLUTION OF SPHERICAL TRIANGLES

**3.51** Right-angled spherical triangles.

$a, b, c$ = sides of triangle, $c$ the side opposite $\gamma$, the right angle.

$\alpha, \beta, \gamma$ = angles opposite $a, b, c$, respectively.

**3.511** Napier's Rules:

The five parts are $a, b, co\ c, co\ \alpha, co\ \beta$, where $co\ c = \frac{\pi}{2} - c$. The right angle $\gamma$ is omitted.

The sine of the middle part is equal to the product of the tangents of the adjacent parts.

The sine of the middle part is equal to the product of the cosines of opposite parts.

From these rules the following equations follow:

$$\sin a = \sin c \sin \alpha,$$
$$\tan a = \tan c \cos \beta = \sin b \tan \alpha,$$
$$\sin b = \sin c \sin \beta,$$
$$\tan b = \tan c \cos \alpha = \sin a \tan \beta,$$
$$\cos \alpha = \cos a \sin \beta,$$
$$\cos \beta = \cos b \sin \alpha,$$
$$\cos c = \cot \alpha \cot \beta = \cos a \cos b.$$

**3.52** Oblique-angled spherical triangles.

$a$, $b$, $c$ = sides of triangle.

$\alpha$, $\beta$, $\gamma$ = angles opposite to $a$, $b$, $c$, respectively.

$s = \frac{1}{2}(a + b + c)$,

$\sigma = \frac{1}{2}(\alpha + \beta + \gamma)$,

$\epsilon = \alpha + \beta + \gamma - 180$ = spherical excess,

$S$ = surface of triangle on sphere of radius $r$.

| *Given* | *Sought* | *Formula* |
|---|---|---|
| $a, b, c$ | $\alpha$ | $\sin^2 \frac{1}{2}\alpha = \text{haversin } \alpha,$ |
| | | $= \dfrac{\sin (s-b) \sin (s-c)}{\sin b \sin c}$ |
| | | $\tan^2 \frac{1}{2}\alpha = \dfrac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}.$ |
| | | $\cos^2 \frac{1}{2}\alpha = \dfrac{\sin s \sin (s-a)}{\sin b \sin c}.$ |
| | | $\text{haversin } \alpha = \dfrac{\text{hav } a - \text{hav } (b-c)}{\sin b \sin c}.$ |
| $\alpha, \beta, \gamma$ | $a$ | $\sin^2 \frac{1}{2}a = \text{haversin } a,$ |
| | | $= \dfrac{-\cos \sigma \cos (\sigma - \alpha)}{\sin \beta \sin \gamma}$ |
| | | $\tan^2 \frac{1}{2}a = \dfrac{-\cos \sigma \cos (\sigma - \alpha)}{\cos (\sigma - \beta) \cos (\sigma - \gamma)}.$ |
| | | $\cos^2 \frac{1}{2}a = \dfrac{\cos (\sigma - \beta) \cos (\sigma - \gamma)}{\sin \beta \sin \gamma}.$ |
| $a, c, \alpha$ Ambiguous case. Two solutions possible. | $\gamma$ | $\sin \gamma = \dfrac{\sin \alpha \sin c}{\sin a}.$ |
| | $\beta$ | $\tan \theta = \tan \alpha \cos c.$ <br> $\sin (\beta + \theta) = \sin \theta \tan c \cot a$ |
| | $b$ | $\cot \phi = \tan c \cos \alpha.$ <br> $\sin (b + \phi) = \dfrac{\cos a \sin \phi}{\cos c}.$ |
| $\alpha, \gamma, c$ Ambiguous case. Two solutions possible. | $c$ | $\sin c = \dfrac{\sin a \sin \gamma}{\sin \alpha}.$ |

| *Given* | *Sought* | *Formula* |
|---|---|---|
| | $b$ | $\tan\theta = \tan a \cos\gamma.$ |
| | | $\sin(b-\theta) = \cot\alpha \tan\gamma \sin\theta.$ |
| | $b$ | $\tan\frac{1}{2}b = \frac{\sin\frac{1}{2}(\alpha+\gamma)}{\sin\frac{1}{2}(\alpha-\gamma)}\tan\frac{1}{2}(a-c)$ |
| | | $= \frac{\cos\frac{1}{2}(\alpha+\gamma)}{\cos\frac{1}{2}(\alpha-\gamma)}\tan\frac{1}{2}(a+c).$ |
| | $\beta$ | $\cot\phi = \cos a \tan\gamma$ |
| | | $\sin(\beta-\phi) = \frac{\cos\alpha\sin\phi}{\cos\gamma}.$ |
| | $\beta$ | $\cot\frac{1}{2}\beta = \frac{\sin\frac{1}{2}(a+c)}{\sin\frac{1}{2}(a-c)}\tan\frac{1}{2}(\alpha-\gamma).$ |
| | | $= \frac{\cos\frac{1}{2}(a+c)}{\cos\frac{1}{2}(a-c)}\tan\frac{1}{2}(\alpha+\gamma).$ |
| $a, b, \gamma$ | $c$ | $\cos c = \cos a \cos b + \sin a \sin b \cos\gamma.$ |
| $\tan\theta = \tan a\cos\gamma$ | | |
| $\tan\phi = \tan b\cos\gamma$ | $c$ | $\cos c = \frac{\cos a\cos(b-\theta)}{\cos\theta}$ |
| | | $= \frac{\cos b\cos(a-\phi)}{\cos\phi}.$ |
| | | $\operatorname{hav} c = \operatorname{hav}(a-b) + \sin a \sin b \operatorname{hav}\gamma$ |
| | $\alpha$ | $\tan\alpha = \frac{\sin\theta\tan\gamma}{\sin(b-\theta)}.$ |
| | $\beta$ | $\sin\beta = \frac{\sin\gamma\sin b}{\sin c}.$ |
| | | $= \frac{\sin\alpha\sin b}{\sin a}.$ |
| | | $\tan\beta = \frac{\sin\phi\tan\gamma}{\sin(a-\phi)}.$ |
| | $\alpha, \beta$ | $\tan\frac{1}{2}(\alpha+\beta) = \frac{\cos\frac{1}{2}(a-b)\cot\frac{1}{2}\gamma}{\cos\frac{1}{2}(a+b)}$ |
| | | $\tan\frac{1}{2}(\alpha-\beta) = \frac{\sin\frac{1}{2}(a-b)\cot\frac{1}{2}\gamma}{\sin\frac{1}{2}(a+b)}.$ |
| $c, \alpha, \beta$ | $\gamma$ | $\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c.$ |
| $\tan\theta = \cos c\tan\alpha$ | | |
| $\tan\phi = \cos c\tan\beta$ | | $\cos\gamma = \frac{\cos\alpha\cos(\beta+\theta)}{\cos\theta}.$ |
| | | $= \frac{\cos\beta\cos(\alpha+\phi)}{\cos\phi}.$ |
| | $a$ | $\tan a = \frac{\tan c\sin\theta}{\sin(\beta+\theta)}.$ |

| *Given* | *Sought* | *Formula* |
|---|---|---|
| | $b$ | $\tan b = \dfrac{\tan c \sin \phi}{\sin (\alpha + \phi)}$. |
| | $a, b$ | $\tan \frac{1}{2}(a+b) = \dfrac{\cos \frac{1}{2}(\alpha - \beta) \tan \frac{1}{2}c}{\cos \frac{1}{2}(\alpha + \beta)}$; $\tan \frac{1}{2}(a-b) = \dfrac{\sin \frac{1}{2}(\alpha - \beta) \tan \frac{1}{2}c}{\sin \frac{1}{2}(\alpha + \beta)}$. |
| $a, b, \gamma$ | $\epsilon$ | $\cot \frac{1}{2}\epsilon = \dfrac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos \gamma}{\sin \gamma}$. |
| $a, b, c$ | $\epsilon$ | $\tan^2 \frac{1}{4}\epsilon = \tan \frac{1}{2}s \tan \frac{1}{2}(s-a) \tan \frac{1}{2}(s-b) \tan \frac{1}{2}(s-c)$. |
| $\epsilon, \gamma$ | $S$ | $S = \dfrac{\epsilon}{180^\circ} \pi r^2$. |

## FINITE SERIES OF CIRCULAR FUNCTIONS

**3.60** If the sum, $f(r)$, of the finite or infinite series:

$$f(r) = a_0 + a_1 r + a_2 r^2 + \ldots\ldots$$

is known, the sums of the series:

$$S_1 = a_0 \cos x + a_1 r \cos (x+y) + a_2 r^2 \cos (x+2y) + \ldots\ldots$$

$$S_2 = a_0 \sin x + a_1 r \sin (x+y) + a_2 r^2 \sin (x+2y) + \ldots\ldots$$

are:

$$S_1 = \tfrac{1}{2}\{e^{ix} f(re^{iy}) + e^{-ix} f(re^{-iy})\},$$

$$S_2 = -\frac{i}{2}\{e^{ix} f(re^{iy}) - e^{-ix} f(re^{-iy})\}.$$

**3.61** Special Finite Series.

1. $$\sum_{k=1}^{n} \sin kx = \frac{\sin \frac{nx}{2} \sin \frac{n+1}{2} x}{\sin \frac{x}{2}}.$$

2. $$\sum_{k=0}^{n} \cos kx = \frac{\cos \frac{nx}{2} \sin \frac{n+1}{2} x}{\sin \frac{x}{2}}.$$

3. $$\sum_{k=1}^{n} \sin^2 kx = \frac{n}{2} - \frac{\cos\ (n+1)x \cdot \sin\ nx}{2\ \sin\ x}.$$

4. $$\sum_{k=0}^{n} \cos^2 kx = \frac{n+2}{2} + \frac{\cos\ (n+1)x \cdot \sin\ nx}{2\ \sin\ x}.$$

5. $$\sum_{k=1}^{n-1} k\ \sin\ kx = \frac{\sin\ nx}{4\ \sin^2 \frac{x}{2}} - \frac{n\ \cos \left(\frac{2n-1}{2}\right)x}{2\ \sin\ \frac{x}{2}}.$$

6. $$\sum_{k=1}^{n-1} k\ \cos\ kx = \frac{n \sin \left(\frac{2n-1}{2}\right)x}{2\ \sin\ \frac{x}{2}} - \frac{1 - \cos\ nx}{4\ \sin^2 \frac{x}{2}}.$$

7. $$\sum_{k=1}^{n} \sin\ (2k-1)x = \frac{\sin^2 nx}{\sin\ x}.$$

8. $$\sum_{k=0}^{n} \sin\ (x+ky) = \frac{\sin \left(x + \frac{ny}{2}\right) \sin \left(\frac{n+1}{2}\, y\right)}{\sin\ \frac{y}{2}}.$$

9. $$\sum_{k=0}^{n} \cos\ (x+ky) = \frac{\cos \left(x + \frac{n}{2}\, y\right) \sin \left(\frac{n+1}{2}\, y\right)}{\sin\ \frac{y}{2}}.$$

10. $$\sum_{k=1}^{n+1} (-1)^{k-1} \sin\ (2k-1)x = (-1)^n\ \frac{\sin\ (2n+2)x}{2\ \cos\ x}.$$

11. $$\sum_{k=1}^{n} (-1)^k \cos\ kx = -\frac{1}{2} + (-1)^n\ \frac{\cos \left(\frac{2n+1}{2}\, x\right)}{2\ \cos\ \frac{x}{2}}.$$

12. $$\sum_{k=1}^{n-1} r^k\ \sin\ kx = \frac{r\ \sin\ x(1 - r^n \cos\ nx) - (1 - r\ \cos\ x) r^n\ \sin\ nx}{1 - 2r\ \cos\ x + r^2}.$$

13. $$\sum_{k=0}^{n-1} r^k\ \cos\ kx = \frac{(1 - r\ \cos\ x)\ (1 - r^n \cos\ nx) + r^{n+1} \sin\ x\ \sin\ nx}{1 - 2r\ \cos\ x + r^2}.$$

14. $$\sum_{k=1}^{n} \left(\frac{1}{2^k} \sec \frac{x}{2^k}\right)^2 = \csc^2 x - \left(\frac{1}{2^n} \csc \frac{x}{2^n}\right)^2.$$

15. $$\sum_{k=1}^{n} \left(2^k \sin^2 \frac{x}{2^k}\right)^2 = \left(2^n \sin \frac{x}{2^n}\right)^2 - \sin^2 x.$$

16. $$\sum_{k=0}^{n} \frac{1}{2^k} \tan \frac{x}{2^k} = \frac{1}{2^n} \cot \frac{x}{2^n} - 2 \cot 2x.$$

17. $$\sum_{k=0}^{n-1} \cos \frac{k^2 2\pi}{n} = \frac{\sqrt{n}}{2}\left(1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right).$$

18. $$\sum_{k=1}^{n-1} \sin \frac{k^2 2\pi}{n} = \frac{\sqrt{n}}{2}\left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right).$$

19. $$\sum_{k=1}^{n-1} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}.$$

20. $$\sum_{k=0}^{n} \frac{1}{2^{2k}} \tan^2 \frac{x}{2^k} = \frac{2^{2n+2} - 1}{3 \cdot 2^{2n-1}} + 4 \cot^2 2x - \frac{1}{2^{2n}} \cot \frac{x}{2^n}.$$

**3.62**

$$S_n = \sum_{k=1}^{n-1} \csc \frac{k\pi}{n}.$$

Watson (Phil. Mag. 31, p. 111, 1916) has obtained an asymptotic expansion for this sum, and has given the following approximation:

$S_n = 2n\{0.7329355992 \log_{10}(2n) - 0.1806453871\}$

$$- \frac{0.087266}{n} + \frac{0.01035}{n^3} - \frac{0.004}{n^5} + \frac{0.005}{n^7} - \cdots$$

Values of $S_n$ are tabulated by integers from $n = 2$ to $n = 30$, and from $n = 30$ to $n = 100$ at intervals of 5.

The expansion of

$$T_n = \sum_{k=1}^{n-1} \csc\left(\frac{k\pi}{n} - \frac{\beta}{2}\right),$$

where $$-\frac{2\pi}{n} < \beta < \frac{2\pi}{n},$$

is also obtained.

**3.70** Finite Products.

1. $$\sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n}{2}-1} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad n \text{ even.}$$

2. $$\cos nx = \prod_{k=1}^{\frac{n}{2}} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{2k-1}{2n}\pi} \right) \quad n \text{ even.}$$

3. $$\sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad n \text{ odd.}$$

4. $$\cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{2k-1}{2n}\pi} \right) \quad n \text{ odd.}$$

5. $$\cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cos x - \cos \left( y + \frac{2k\pi}{n} \right) \right\}.$$

6. $$a^{2n} - 2a^n b^n \cos nx + b^{2n} = \prod_{k=0}^{n-1} \left\{ a^2 - 2ab \cos \left( x + \frac{2k\pi}{n} \right) + b^2 \right\}.$$

ROOTS OF TRANSCENDENTAL EQUATIONS

**3.800** $\tan x = x$.

The first 17 roots, and the corresponding maxima and minima of $\frac{\sin x}{x}$ are given in the following table (Lommel, Abh. Munch. Akad. (2) 15, 123, 1886):

| $n$ | $x_n$ | Max $\frac{\sin x}{x}$ Min |
|---|---|---|
| 1 | 0 | 1 |
| 2 | 4.4934 | −0.2172 |
| 3 | 7.7253 | +0.1284 |
| 4 | 10.9041 | −0.0913 |
| 5 | 14.0662 | +0.0709 |
| 6 | 17.2208 | −0.0580 |
| 7 | 20.3713 | +0.0490 |
| 8 | 23.5195 | −0.0425 |
| 9 | 26.6661 | +0.0375 |
| 10 | 29.8116 | −0.0335 |
| 11 | 32.9564 | +0.0303 |
| 12 | 36.1006 | −0.0277 |
| 13 | 39.2444 | +0.0255 |
| 14 | 42.3879 | −0.0236 |
| 15 | 45.5311 | +0.0220 |
| 16 | 48.6741 | −0.0205 |
| 17 | 51.8170 | +0.0193 |

**3.801**

$$\tan x = \frac{2x}{2 - x^2}.$$

The first three roots are:

$$x_1 = 0,$$

$$x_2 = 119.26\ \frac{\pi}{180},$$

$$x_3 = 340.35\ \frac{\pi}{180}.$$

If $x$ is large

$$x_n = n\pi - \frac{2}{n\pi} - \frac{16}{3n^3\pi^3} + \ldots.$$

(Rayleigh, Theory of Sound, II, p. 265.)

**3.802**

$$\tan x = \frac{x^3 - 9x}{4x^2 - 9}.$$

The first two roots are:

$$x_1 = 0,$$

$$x_2 = 3.3422.$$

(Rayleigh, l. c. p. 266.)

**3.803**

$$\tan x = \frac{x}{1 - x^2}.$$

The first two roots are:

$$x_1 = 0,$$

$$x_2 = 2.744.$$

(J. J. Thomson, Recent Researches, p. 373.)

**3.804**

$$\tan x = \frac{3x}{3 - x^2}.$$

The first seven roots are:

$$x_1 = 0,$$

$$x_2 = 1.8346\pi,$$

$$x_3 = 2.8950\pi,$$

$$x_4 = 3.9225\pi,$$

$$x_5 = 4.9385\pi,$$

$$x_6 = 5.9489\pi,$$

$$x_7 = 6.9563\pi.$$

(Lamb, London Math. Soc. Proc. 13, 1882.)

**3.805**

$$\tan x = \frac{4x}{4 - 3x^2}.$$

The first seven roots are:

$$x_1 = 0,$$
$$x_2 = 0.8160\pi,$$
$$x_3 = 1.9285\pi,$$
$$x_4 = 2.9359\pi,$$
$$x_5 = 3.9658\pi,$$
$$x_6 = 4.9728\pi,$$
$$x_7 = 5.9774\pi.$$

(Lamb, l. c.)

**3.806**

$$\cos x \cosh x = 1.$$

The roots are:

$$x_1 = 4.7300408,$$
$$x_2 = 7.8532046,$$
$$x_3 = 10.9956078,$$
$$x_4 = 14.1371655,$$
$$x_5 = 17.2787596,$$
$$x_n = \tfrac{1}{2}(2n+1)\pi \quad n>5.$$

(Rayleigh, Theory of Sound, I, p. 278.)

**3.807**

$$\cos x \cosh x = -1.$$

The roots are:

$$x_1 = 1.875104,$$
$$x_2 = 4.694098,$$
$$x_3 = 7.854757,$$
$$x_4 = 10.995541,$$
$$x_5 = 14.137168,$$
$$x_6 = 17.278759,$$
$$x_n = \tfrac{1}{2}(2n-1)\pi \quad n>6.$$

**3.808**

$$1 - (1 + x^2)\cos x = 0.$$

The roots are:

$$x_1 = 1.102506,$$
$$x_2 = 4.754761,$$
$$x_3 = 7.837964,$$
$$x_4 = 11.003766,$$
$$x_5 = 14.132185,$$
$$x_6 = 17.282097.$$

(Schlömilch: Ubungsbuch, I, p. 354.)

**3.809** The smallest root of

$$\theta - \cot\theta = 0,$$

is

$$\theta = 49^\circ\ 17'\ 36''.5.$$

(l. c. p. 355.)

**3.810** The smallest root of

$$\theta - \cos\theta = 0,$$

is

$$\theta = 42^\circ\ 20'\ 47''.3.$$

(l. c. p. 353.)

**3.811** The smallest root of

$$xe^x - 2 = 0,$$

is

$$x = 0.8526.$$

(l. c. p. 353.)

**3.812** The smallest root of

$$\log(1 + x) - \tfrac{3}{4}x = 0,$$

is

$$x = 0.73360.$$

(l. c. p. 353.)

**3.813**

$$\tan x - x + \frac{1}{x} = 0.$$

The first roots are:

$$x_1 = 4.480,$$
$$x_2 = 7.723,$$
$$x_3 = 10.90,$$
$$x_4 = 14.07.$$

(Collo, Annalen der Physik, 65, p. 45, 1921.)

**3.814**

$$\cot x + x - \frac{1}{x} = 0.$$

The first roots are:

$$x_1 = 0,$$
$$x_2 = 2.744,$$
$$x_3 = 6.117,$$
$$x_4 = 9.317,$$
$$x_5 = 12.48,$$
$$x_6 = 15.64,$$
$$x_7 = 18.80.$$

(Collo, l. c.)

**3.90** Special Tables.

$\sin\theta$, $\cos\theta$: The British Association Report for 1916 contains the following tables:

Table I, p. 60. $\sin\theta$, $\cos\theta$, $\theta$ expressed in radians from $\theta = 0$ to $\theta = 1.600$, interval 0.001, 10 decimal places.

Table II, p. 88. $\theta - \sin\theta$, $1 - \cos\theta$, $\theta = 0.00001$ to $\theta = 0.00100$, interval 0.00001, 10 decimal places.

Table III, p. 90. $\sin\theta$, $\cos\theta$; $\theta = 0.1$ to $\theta = 10.0$, interval 0.1, 15 decimal places.

J. Peters (Abh. d. K. P. Akad. der Wissen., Berlin, 1911) has given sines and cosines for every sexagesimal second to 21 places.

hav $\theta$, $\log_{10}$ hav $\theta$: Bowditch, American Practical Navigator, five-place tables, $0° - 180°$, for 15″ intervals.

Tables for Solution of Spherical Triangles.

Aquino's Altitude and Azimuth Tables, London, 1918. Reprinted in Hydrographic Office Publication, No. 200, Washington, 1918.

Hyperbolic Functions.

The Smithsonian Mathematical Tables: Hyperbolic Functions, contain the most complete five-place tables of Hyperbolic Functions.

Table I. The common logarithms (base 10) of $\sinh u$, $\cosh u$, $\tanh u$, $\coth u$:

$u = 0.0001$ to $u = 0.1000$ interval 0.0001,
$u = 0.001$ to $u = 3.000$ interval 0.001,
$u = 3.00$ to $u = 6.00$ interval 0.01.

Table II. $\sinh u$, $\cosh u$, $\tanh u$, $\coth u$. Same ranges and intervals.

Table III. $\sin u$, $\cos u$, $\log_{10}\sin u$, $\log_{10}\cos u$:

$u = 0.0001$ to $u = 0.1000$ interval 0.0001,
$u = 0.100$ to $u = 1.600$ interval 0.001.

Table IV. $\log_{10}e^u$ (7 places), $e^u$ and $e^{-u}$ (7 significant figures):

$u = 0.001$ to $u = 2.950$ interval 0.001,
$u = 3.00$ to $u = 6.00$ interval 0.01,
$u = 1.0$ to $u = 100$ interval 1.0 (9–10 figures).

Table V. five-place table of natural logarithms, $\log u$.

$u = 1.0$ to $u = 1000$ interval 1.0,
$u = 1000$ to $u = 10{,}000$ varying intervals.

Table VI. $gd\ u$ (7 places); $u$ expressed in radians, $u = 0.001$ to $u = 3.000$, interval 0.001, and the corresponding angular measure. $u = 3.00$ to $u = 6.00$, interval 0.01.

Table VII. $gd\ ^1u$, to $0'.01$, in terms of $gd\ u$ in degrees and minutes from $0°\ 1'$ to $89°\ 59'$.

Table VIII. Table for conversion of radians into angular measure.

Kennelly: Tables of Complex Hyperbolic and Circular Functions. Cambridge, Harvard University Press, 1914.

The complex argument, $x + iq = \rho e^{i\delta}$. In the tables this is denoted $\rho \angle \delta$. $\rho = \sqrt{x^2 + q^2}$, $\tan \delta = q/x$.

Tables I, II, III give the hyperbolic sine, cosine and tangent of $(\rho \angle \delta)$ expressed as $r \angle \gamma$:

$\delta = 45°$ to $\delta = 90°$ interval 1°

$\rho = 0.01$ to $\rho = 3.0$ interval 0.1.

Tables IV and V give $\frac{\sinh \theta}{\theta}$, $\frac{\tanh \theta}{\theta}$ expressed as $r \angle \gamma$, $\theta = \rho \angle \delta$,

$\rho = 0.1$ to $\rho = 3.0$ interval 0.1,

$\delta = 45°$ to $\delta = 90°$ interval 1°.

Table VI gives $\sinh (\rho \angle 45°)$, $\cosh (\rho \angle 45°)$, $\tanh (\rho \angle 45°)$, $\coth (\rho \angle 45°)$, $\operatorname{sech} (\rho \angle 45°)$, $\operatorname{csch} (\rho \angle 45°)$ expressed as $r \angle \gamma$:

$\rho = 0$ to $\rho = 6.0$ interval 0.1,

$\rho = 6.05$ to $\rho = 20.50$ interval 0.05.

Tables VII, VIII and IX give $\sinh (x + iq)$, $\cosh (x + iq)$, $\tanh (x + iq)$, expressed as $u + iv$:

$x = 0$ to $x = 3.95$ interval 0.05,

$q = 0$ to $q = 2.0$ interval 0.05.

Tables X, XI, XII give $\sinh (x + iq)$, $\cosh (x + iq)$, $\tanh (x + iq)$ expressed as $r \angle \gamma$:

$x = 0$ to $x = 3.95$ interval 0.05,

$q = 0$ to $q = 2.0$ interval 0.05.

Table XIII gives $\sinh (4 + iq)$, $\cosh (4 + iq)$, $\tanh (4 + iq)$ expressed both as $u + iv$ and $r \angle \gamma$:

$q = 0$ to $q = 2.0$ interval 0.05.

Table XIV gives $\frac{e^x}{2}$ and $\log_{10} \frac{e^x}{2}$.

$x = 4.00$ to $x = 10.00$ interval 0.01.

Table XV gives the real hyperbolic functions: $\sinh \theta$, $\cosh \theta$, $\tanh \theta$, $\coth \theta$, $\operatorname{sech} \theta$, $\operatorname{csch} \theta$.

$\theta = 0$ to $\theta = 2.5$ interval 0.01,

$\theta = 2.5$ to $\theta = 7.5$ interval 0.1.

Pernot and Woods: Logarithms of Hyperbolic Functions to 12 Significant Figures. Berkeley, University of California Press, 1918.

Table I. $\log_{10} \sinh x$, with the first three differences.

$x = .0000$ to $x = 2\ 018$ nterval 0.001.

Table II. $\log_{10} \cosh x$.

$x = 0.000$ to $x = 2.032$ interval 0.001.

Table III. $\log_{10} \tanh x$.

$x = 0.000$ to $x = 2.018$ interval 0.001.

Table IV. $\log_{10} \frac{\sinh x}{x}$.

$x = 0.00$ to $x = 0.506$ interval 0.001.

Table V. $\log_{10} \frac{\tanh x}{x}$.

$x = 0.000$ to $x = 0.506$ interval 0.001.

Van Orstrand, Memoirs of the National Academy of Sciences, Vol. XIV, fifth memoir, Washington, 1921.

Tables of $\frac{1}{n!}$, $e^x$, $e^{-x}$, $e^{n\pi}$, $e^{-n\pi}$, $e^{\pm\frac{n\pi}{360}}$, $\sin x$, $\cos x$, to 23–62 decimal places or significant figures.

# IV. VECTOR ANALYSIS

**4.000** A vector **A** has components along the three rectangular axes, $x, y, z$: $A_x, A_y, A_z$.

$$A = \text{length of vector.}$$
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

Direction cosines of **A**, $\frac{A_x}{A}, \frac{A_y}{A}, \frac{A_z}{A}$.

**4.001** Addition of vectors.

$$\mathbf{A} + \mathbf{B} = \mathbf{C}.$$

**C** is a vector with components.

$$C_x = A_x + B_x.$$
$$C_y = A_y + B_y.$$
$$C_z = A_z + B_z.$$

**4.002** $\theta$ = angle between **A** and **B**.

$$C = \sqrt{A^2 + B^2 + 2AB\cos\theta}.$$
$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}.$$

**4.003** If **a**, **b**, **c** are any three non-coplanar vectors of unit length, any vector, **R**, may be expressed:

$$\mathbf{R} = a\mathbf{a} + b\mathbf{b} + c\mathbf{c},$$

where $a, b, c$ are the lengths of the projections of **R** upon **a**, **b**, **c** respectively.

**4.004** Scalar product of two vectors:

$$S\mathbf{AB} = (\mathbf{AB}) = \mathbf{AB}$$

are equivalent notations.

$$\mathbf{AB} = AB\cos\widehat{AB}.$$

**4.005** Vector product of two vectors:

$$V\mathbf{AB} = \mathbf{A} \times \mathbf{B} = [\mathbf{AB}] = \mathbf{C}.$$

**C** is a vector whose length is

$$C = AB\sin\widehat{AB}.$$

The direction of **C** is perpendicular to both **A** and **B** such that a right-handed rotation about **C** through the angle $\widehat{AB}$ turns **A** into **B**.

**4.006** $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three unit vectors perpendicular to each other. If their directions coincide with the axes $x, y, z$ of a rectangular system of coördinates:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

**4.007**

$$\mathbf{ii} = \mathbf{i}^2 = \mathbf{jj} = \mathbf{j}^2 = \mathbf{kk} = \mathbf{k}^2 = 1,$$
$$\mathbf{ij} = \mathbf{ji} = \mathbf{jk} = \mathbf{kj} = \mathbf{ki} = \mathbf{ik} = 0.$$

**4.008**

$$V\mathbf{ij} = -V\mathbf{ji} = \mathbf{k},$$
$$V\mathbf{jk} = -V\mathbf{kj} = \mathbf{i},$$
$$V\mathbf{ki} = -V\mathbf{ik} = \mathbf{j}.$$

**4.009**

$$\mathbf{AB} = \mathbf{BA} = AB\cos\widehat{AB} = A_xB_x + A_yB_y + A_zB_z.$$

**4.010**

$$V\mathbf{AB} = -V\mathbf{BA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}.$$

**4.10** If $\mathbf{A}, \mathbf{B}, \mathbf{C}$, are any three vectors:

$$\mathbf{A}V\mathbf{BC} = \mathbf{B}V\mathbf{CA} = \mathbf{C}V\mathbf{AB}$$
$$= \text{Volume of parallelepipedon having } \mathbf{A}, \mathbf{B}, \mathbf{C} \text{ as edges}$$
$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

**4.11**

1. $V\mathbf{A}(\mathbf{B} + \mathbf{C}) = V\mathbf{AB} + V\mathbf{AC}.$
2. $V(\mathbf{A} + \mathbf{B})(\mathbf{C} + \mathbf{D}) = V\mathbf{A}(\mathbf{C} + \mathbf{D}) + V\mathbf{B}(\mathbf{C} + \mathbf{D}).$
3. $V\mathbf{A}V\mathbf{BC} = \mathbf{B}S\mathbf{AC} - \mathbf{C}S\mathbf{AB}.$
4. $V\mathbf{A}V\mathbf{BC} + V\mathbf{B}V\mathbf{CA} + V\mathbf{C}V\mathbf{AB} = 0.$
5. $V\mathbf{AB}\cdot V\mathbf{CD} = \mathbf{AC}\cdot\mathbf{BD} - \mathbf{BC}\cdot\mathbf{AD}.$
6. $V(V\mathbf{AB}\cdot V\mathbf{CD}) = \mathbf{C}S(\mathbf{D}V\mathbf{AB}) - \mathbf{D}S(\mathbf{C}V\mathbf{AB})$
$= \mathbf{C}S(\mathbf{A}V\mathbf{BD}) - \mathbf{D}S(\mathbf{A}V\mathbf{BC})$
$= \mathbf{B}S(\mathbf{A}V\mathbf{CD}) - \mathbf{A}S(\mathbf{B}V\mathbf{CD})$
$= \mathbf{B}S(\mathbf{C}V\mathbf{DA}) - \mathbf{A}S(\mathbf{C}V\mathbf{DB}).$

**4.20**

1. $$d\mathbf{AB} = \mathbf{A}d\,\mathbf{B} + \mathbf{B}d\mathbf{A}.$$
2. $$\begin{aligned} dV\mathbf{AB} &= V\mathbf{A}d\mathbf{B} + Vd\mathbf{AB} \\ &= V\mathbf{A}d\mathbf{B} - V\mathbf{B}d\mathbf{A}. \end{aligned}$$

**4.21**

1. $$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$
2. $$\nabla\mathbf{A} = \operatorname{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$
3. $$\nabla\phi = \operatorname{grad}\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}.$$
4. $$\begin{aligned} V\nabla\mathbf{A} &= \operatorname{curl}\mathbf{A} = \operatorname{rot}\mathbf{A} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \end{aligned}$$
5. $$\nabla\nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

**4.22**

1. $\operatorname{curl}\operatorname{grad}\phi = \operatorname{curl}\nabla\phi = V\nabla\nabla\phi = 0.$
2. $$\operatorname{div}\operatorname{grad}\phi = \nabla\nabla\phi = \overline{\nabla}^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$
3. $\operatorname{div}\operatorname{curl}\mathbf{A} = 0.$
4. $\operatorname{curl}\operatorname{curl}\mathbf{A} = \operatorname{curl}^2\mathbf{A} = \nabla\operatorname{div}\mathbf{A} - \overline{\nabla}^2\mathbf{A}.$
5. $$\overline{\nabla}^2\mathbf{A} = \mathbf{i}\overline{\nabla}^2\mathbf{A}_x + \mathbf{j}\overline{\nabla}^2 A_y + \mathbf{k}\overline{\nabla}^2 A_z.$$
6. $$\mathbf{A}\nabla = A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}.$$

**4.23**

1. $\nabla \mathbf{AB} = \text{grad } \mathbf{AB} = (\mathbf{A}\nabla)\mathbf{B} + (\mathbf{B}\nabla)\mathbf{A} + V.\mathbf{A} \text{ curl } \mathbf{B} + V.\mathbf{B} \text{ curl } \mathbf{A}.$
2. $\nabla V\mathbf{AB} = \text{div } V\mathbf{AB} = \mathbf{B} \text{ curl } \mathbf{A} - \mathbf{A} \text{ curl } \mathbf{B}.$
3. $V\nabla V\mathbf{AB} = (\mathbf{B}\nabla)\mathbf{A} - (\mathbf{A}\nabla)\mathbf{B} + \mathbf{A} \text{ div } B - \mathbf{B} \text{ div } \mathbf{A}.$
4. $\text{div } \phi\mathbf{A} = \phi \text{ div } \mathbf{A} + \mathbf{A}\nabla\phi.$
5. $\text{curl } \phi\mathbf{A} = V \cdot \nabla\phi\mathbf{A} + \phi \text{ curl } \mathbf{A} = V \cdot \text{grad } \phi.\mathbf{A} + \phi \text{ curl } \mathbf{A}.$
6. $\nabla\mathbf{A}^2 = 2(\mathbf{A}\nabla)\mathbf{A} + 2V\mathbf{A} \text{ curl } \mathbf{A}.$
7. $\mathbf{C}(\mathbf{A}\nabla)\mathbf{B} = \mathbf{A}(\mathbf{C}\nabla)\mathbf{B} + \mathbf{A}V\mathbf{C} \text{ curl } \mathbf{B}.$
8. $\mathbf{B}\nabla\mathbf{A}^2 = 2\mathbf{A}(\mathbf{B}\nabla)\mathbf{A}.$

**4.24** $\mathbf{R}$ is a radius vector of length $r$ and $\mathbf{r}$ a unit vector in the direction of $\mathbf{R}$.

$$\mathbf{R} = r\mathbf{r},$$
$$r^2 = x^2 + y^2 + z^2.$$

1. $$\nabla \frac{1}{r} = -\frac{1}{r^3}\mathbf{R} = -\frac{1}{r^2}\mathbf{r}.$$
2. $$\nabla^2 \frac{1}{r} = 0.$$
3. $$\nabla r = \frac{1}{r}\mathbf{R} = \mathbf{r} = \text{grad } r.$$
4. $$\nabla^2 r = \frac{2}{r}.$$
5. $$V\nabla\mathbf{R} = \text{curl } \mathbf{R} = 0.$$
6. $$\nabla\mathbf{R} = \text{div } \mathbf{R} = 3.$$
7. $$\frac{d\phi}{dr} = \mathbf{r}\nabla\phi.$$
8. $$(\mathbf{R}\nabla)\mathbf{A} = r\frac{d\mathbf{A}}{dr}.$$
9. $$(\mathbf{r}\nabla)\mathbf{A} = \frac{d\mathbf{A}}{dr}.$$
10. $$(\mathbf{A}\nabla)\mathbf{R} = \mathbf{A}.$$

**4.30** $d\mathbf{S}$ = an element of area of a surface regarded as a vector whose direction is that of the positive normal to the surface.

$dV$ = an element of volume — a scalar.

$d\mathbf{s}$ = an element of arc of a curve regarded as a vector whose direction is that of the positive tangent to the curve.

**4.31** Gauss's Theorem:

$$\int\int\int \text{div } \mathbf{A}\,dV = \int\int \mathbf{A}\,d\mathbf{S}.$$

**4.32** Green's Theorem:

1. $\int\int\int \phi\nabla^2\psi\,dV + \int\int\int \nabla\phi\nabla\psi\,dV = \int\int \phi\nabla\psi\,d\mathbf{S}$
2. $\int\int\int (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \int\int (\phi\nabla\psi - \psi\nabla\phi)\,d\mathbf{S}.$

**4.33** Stokes's Theorem:

$$\int\int \text{curl } \mathbf{A}\,d\mathbf{S} = \int \mathbf{A}\,d\mathbf{s}.$$

**4.40** A polar vector is one whose components, referred to a rectangular system of axes, all change in sign when the three axes are reversed.

**4.401** An axial vector is one whose components are unchanged when the axes are reversed.

**4.402** The vector product of two polar or of two axial vectors is an axial vector.

**4.403** The vector product of a polar and an axial vector is a polar vector.

**4.404** The curl of a polar vector is an axial vector and the curl of an axial vector is a polar vector.

**4.405** The scalar product of two polar or of two axial vectors is a true scalar, i.e., it keeps its sign if the axes to which the vectors are referred are reversed.

**4.406** The scalar product of an axial vector and a polar vector is a pseudo-scalar, i.e., it changes in sign when the axes of reference are reversed.

**4.407** The product or quotient of a polar vector and a true scalar is a polar vector; of an axial vector and a true scalar an axial vector; of a polar vector and a pseudo-scalar an axial vector; of an axial vector and a pseudo-scalar a polar vector.

**4.408** The gradient of a true scalar is a polar vector; the gradient of a pseudo-scalar is an axial vector.

**4.409** The divergence of a polar vector is a true scalar; of an axial vector a pseudo-scalar.

## 4.6 Linear Vector Functions.

**4.610** A vector **Q** is a linear vector function of a vector **R** if its components, $Q_1, Q_2, Q_3$, along any three non-coplanar axes are linear functions of the components $R_1, R_2, R_3$ of **R** along the same axes.

**4.611** Linear Vector Operator. If $\hat{\omega}$ is the linear vector operator,

$$\mathbf{Q} = \hat{\omega}\mathbf{R}.$$

This is equivalent to the three scalar equations,

$$\begin{aligned} Q_1 &= \omega_{11}R_1 + \omega_{12}R_2 + \omega_{13}R_3, \\ Q_2 &= \omega_{21}R_1 + \omega_{22}R_2 + \omega_{23}R_3, \\ Q_3 &= \omega_{31}R_1 + \omega_{32}R_2 + \omega_{33}R_3. \end{aligned}$$

**4.612** If **a, b, c** are the three non-coplanar unit axes,

$$\begin{aligned} \omega_{11} &= S.\mathbf{a}\hat{\omega}\mathbf{a}, & \omega_{21} &= S.\mathbf{b}\hat{\omega}\mathbf{a}, & \omega_{31} &= S.\mathbf{c}\hat{\omega}\mathbf{a}, \\ \omega_{12} &= S.\mathbf{a}\hat{\omega}\mathbf{b}, & \omega_{22} &= S.\mathbf{b}\hat{\omega}\mathbf{b}, & \omega_{32} &= S.\mathbf{c}\hat{\omega}\mathbf{b}, \\ \omega_{13} &= S.\mathbf{a}\hat{\omega}\mathbf{c}, & \omega_{23} &= S.\mathbf{b}\hat{\omega}\mathbf{c} & \omega_{33} &= S.\mathbf{c}\hat{\omega}\mathbf{c}. \end{aligned}$$

**4.613** The conjugate linear vector operator $\hat{\omega}'$ is obtained from $\hat{\omega}$ by replacing $\omega_{hk}$ by $\omega_{kh}$; $h, k = 1, 2, 3$.

**4.614** In the symmetrical, or self-conjugate linear vector operator, denoted by $\omega$,

$$\omega = \tfrac{1}{2}(\hat{\omega} + \hat{\omega}').$$

Hence by **4.612**

$$S.\mathbf{a}\omega\mathbf{b} = S.\mathbf{b}\omega\mathbf{a}, \text{ etc.}$$

**4.615** The general linear vector function $\hat{\omega}\mathbf{R}$ may always be resolved into the sum of a self-conjugate linear vector function of **R** and the vector product of **R** by a vector **c:**

$$\hat{\omega}\mathbf{R} = \omega\mathbf{R} + V.\mathbf{c}\mathbf{R},$$

where

$$\omega = \tfrac{1}{2}(\hat{\omega} + \hat{\omega}'),$$

and

$$\mathbf{c} = \tfrac{1}{2}(\omega_{32} - \omega_{23})\mathbf{i} + \tfrac{1}{2}(\omega_{13} - \omega_{31})\mathbf{j} + \tfrac{1}{2}(\omega_{21} - \omega_{12})\mathbf{k},$$

if **i, j, k** are three mutually perpendicular unit vectors.

**4.616** The general linear vector operator $\hat{\omega}$ may be determined by three non-coplanar vectors, **A, B, C,** where,

$$\mathbf{A} = \mathbf{a}\omega_{11} + \mathbf{b}\omega_{12} + \mathbf{c}\omega_{13},$$
$$\mathbf{B} = \mathbf{a}\omega_{21} + \mathbf{b}\omega_{22} + \mathbf{c}\omega_{23},$$
$$\mathbf{C} = \mathbf{a}\omega_{31} + \mathbf{b}\omega_{32} + \mathbf{c}\omega_{33},$$

and

$$\hat{\omega} = \mathbf{a}S.\mathbf{A} + \mathbf{b}S.\mathbf{B} + \mathbf{c}S.\mathbf{C}.$$

**4.617** If $\hat{\omega}$ is the general linear vector operator and $\hat{\omega}'$ its conjugate,

$$\hat{\omega}\mathbf{R} = \mathbf{R}\hat{\omega}',$$
$$\hat{\omega}'\mathbf{R} = \mathbf{R}\hat{\omega}$$

**4.620** The symmetrical or self-conjugate linear vector operator has three mutually perpendicular axes. If these be taken along **i, j, k,**

$$\omega = \mathbf{i}S.\omega_1\mathbf{i} + \mathbf{j}S.\omega_2\mathbf{j} + \mathbf{k}S.\omega_3\mathbf{k},$$

where $\omega_1$, $\omega_2$, $\omega_3$ are scalar quantities, the principal values of $\omega$.

**4.621** Referred to any system of three mutually perpendicular unit vectors, **a, b, c,** the self-conjugate operator, $\omega$, is determined by the three vectors (**4.616**):

$$\mathbf{A} = \omega\mathbf{a} = \mathbf{a}\omega_{11} + \mathbf{b}\omega_{12} + \mathbf{c}\omega_{13},$$
$$\mathbf{B} = \omega\mathbf{b} = \mathbf{a}\omega_{21} + \mathbf{b}\omega_{22} + \mathbf{c}\omega_{23},$$
$$\mathbf{C} = \omega\mathbf{c} = \mathbf{a}\omega_{31} + \mathbf{b}\omega_{32} + \mathbf{c}\omega_{33},$$

where

$$\omega_{hk} = \omega_{kh},$$
$$\omega = \mathbf{a}S.\mathbf{A} + \mathbf{b}S.\mathbf{B} + \mathbf{c}S.\mathbf{C}.$$

**4.622** If $n$ is one of the principal values, $\omega_1$, $\omega_2$, $\omega_3$, these are given by the roots of the cubic,

$$n^3 - n^2(S.\mathbf{Aa} + S.\mathbf{Bb} + S.\mathbf{Cc}) + n(S.\mathbf{a}V\mathbf{BC} + S.\mathbf{b}V\mathbf{CA} + \mathbf{S.c}V\mathbf{A}B) - S.\mathbf{A}V\mathbf{BC} = 0.$$

**4.623** In transforming from one to another system of rectangular axes the following are invariant:

$$S.\mathbf{Aa} + S.\mathbf{Bb} + S.\mathbf{Cc} = \omega_1 + \omega_2 + \omega_3.$$
$$S\mathbf{a}V\mathbf{BC} + S.\mathbf{b}V\mathbf{CA} + S.\mathbf{c}V\mathbf{AB} = \omega_2\omega_3 + \omega_3\omega_1 + \omega_1\omega_2.$$
$$S.\mathbf{A}V\mathbf{BC} = \omega_1\omega_2\omega_3.$$

**4.624**

$$\omega_1 + \omega_2 + \omega_3 = \omega_{11} + \omega_{22} + \omega_{33},$$
$$\omega_2\omega_3 + \omega_3\omega_1 + \omega_1\omega_2 = \omega_{22}\omega_{33} + \omega_{33}\omega_{11} + \omega_{11}\omega_{22} - \omega^2_{23} - \omega^2_{31} + \omega^2_{12},$$
$$\omega_1\omega_2\omega_3 = \omega_{11}\omega_{22}\omega_{33} + 2\omega_{23}\omega_{31}\omega_{12} - \omega_{11}\omega^2_{23} - \omega_{22}\omega^2_{31} - \omega_{33}\omega^2_{12}.$$

**4.625** The principal axes of the self-conjugate operator, $\omega$, are those of the quadric:

$$\omega_{11}x^2 + \omega_{22}y^2 + \omega_{33}z^2 + 2\omega_{23}yz + 2\omega_{31}zx + 2\omega_{12}xy = \text{const.},$$

where $x$, $y$, $z$ are rectangular axes in the direction of **a, b, c** respectively.

**4.626** Referred to its principal axes the equation of the quadric is,

$$\omega_1 x^2 + \omega_2 y^2 + \omega_3 z^2 = \text{const.}$$

**4.627** Applying the self-conjugate operator, $\omega$, successively,

$$\begin{aligned}
\omega\mathbf{R} &= \mathbf{i}\omega_1 R_1 + \mathbf{j}\omega_2 R_2 + \mathbf{k}\omega_3 R_3,\\
\omega\omega\mathbf{R} &= \omega^2\mathbf{R} = \omega_1^2 R_1 + \mathbf{j}\omega_2^2 R_2 + \mathbf{k}\omega_3^2 R_3,\\
\omega\omega^2\mathbf{R} &= \omega^3\mathbf{R} = \mathbf{i}\omega_1^3 R_1 + \mathbf{j}\omega_2^3 R_2 + \mathbf{k}\omega_3^3 R_3,\\
&\ldots\\
&\ldots\\
\omega^{-1}\mathbf{R} &= \mathbf{i}\frac{R_1}{\omega_1} + \mathbf{j}\frac{R_2}{\omega_2} + \mathbf{k}\frac{R_3}{\omega_3}.\\
&\ldots\\
&\ldots
\end{aligned}$$

**4.628** Applying a number of self-conjugate operators, $\alpha$, $\beta$, . . . ., all with the same axes but with different principal values $(\alpha_1\alpha_2\alpha_3)$, $(\beta_1\beta_2\beta_3)$, . . . .

$$\begin{aligned}
\alpha\mathbf{R} &= \mathbf{i}\alpha R_1 + \mathbf{j}\alpha_2 R_2 + \mathbf{k}\alpha_3 R_3,\\
\beta\alpha\mathbf{R} &= \alpha\beta\mathbf{R} = \mathbf{i}\alpha_1\beta_1 R_1 + \mathbf{j}\alpha_2\beta_2 R_2 + \mathbf{k}\alpha_3\beta_3 R_3.\\
&\ldots
\end{aligned}$$

**4.629**

$$\begin{aligned}
S.\mathbf{Q}\omega\mathbf{R} &= S.\mathbf{R}\omega Q,\\
&= \omega_1 Q_1 R_1 + \omega_2 Q_2 R_2 + \omega_3 Q_3 R_3.
\end{aligned}$$

# V. CURVILINEAR COÖRDINATES

**5.00** Given three surfaces.

1.
$$\begin{cases} u = f_1(x, y, z), \\ v = f_2(x, y, z), \\ w = f_3(x, y, z). \end{cases}$$

2.
$$\begin{cases} x = \phi_1(u, v, w), \\ y = \phi_2(u, v, w), \\ z = \phi_3(u, v, w). \end{cases}$$

3.
$$\begin{cases} \frac{1}{h_1^2} = \left(\frac{\partial \phi_1}{\partial u}\right)^2 + \left(\frac{\partial \phi_2}{\partial u}\right)^2 + \left(\frac{\partial \phi_3}{\partial u}\right)^2, \\ \frac{1}{h_2^2} = \left(\frac{\partial \phi_1}{\partial v}\right)^2 + \left(\frac{\partial \phi_2}{\partial v}\right)^2 + \left(\frac{\partial \phi_3}{\partial v}\right)^2, \\ \frac{1}{h_3^2} = \left(\frac{\partial \phi_1}{\partial w}\right)^2 + \left(\frac{\partial \phi_2}{\partial w}\right)^2 + \left(\frac{\partial \phi_3}{\partial w}\right)^2. \end{cases}$$

4.
$$\begin{cases} g_1 = \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_1}{\partial w} + \frac{\partial \phi_2}{\partial v}\frac{\partial \phi_2}{\partial w} + \frac{\partial \phi_3}{\partial v}\frac{\partial \phi_3}{\partial w}, \\ g_2 = \frac{\partial \phi_1}{\partial w}\frac{\partial \phi_1}{\partial u} + \frac{\partial \phi_2}{\partial w}\frac{\partial \phi_2}{\partial u} + \frac{\partial \phi_3}{\partial w}\frac{\partial \phi_3}{\partial u}, \\ g_3 = \frac{\partial \phi_1}{\partial u}\frac{\partial \phi_1}{\partial v} + \frac{\partial \phi_2}{\partial u}\frac{\partial \phi_2}{\partial v} + \frac{\partial \phi_3}{\partial u}\frac{\partial \phi_3}{\partial v}. \end{cases}$$

**5.01** The linear element of arc, $ds$, is given by:

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{du^2}{h_1^2} + \frac{dv^2}{h_2^2} + \frac{dw^2}{h_3^2} + 2g_1\, dv\, dw + 2g_2\, dw\, du + 2g_3\, du\, dv.$$

**5.02** The surface elements, areas of parallelograms on the three surfaces, are:

$$dS_u = \frac{dv\, dw}{h_2 h_3}\sqrt{1 - h_2^2 h_3^2 g_1^2},$$

$$dS_v = \frac{dw\, du}{h_3 h_1}\sqrt{1 - h_3^2 h_1^2 g_2^2},$$

$$dS_w = \frac{du\, dv}{h_1 h_2}\sqrt{1 - h_1^2 h_2^2 g_3^2}.$$

**5.03** The volume of an elementary parallelepipedon is:

$$d\tau = \frac{du\,dv\,dw}{h_1h_2h_3}\left\{1 - h_1^2h_2^2g_3^2 - h_2^2h_3^2g_1^2 - h_3^2h_1^2g_2^2 + h_1^2h_2^2h_3^2g_1g_2g_3\right\}^{\frac{1}{2}}$$

**5.04** $\omega_1$, $\omega_2$, $\omega_3$ are the angles between the normals to the surface $f_2$, $f_3$; $f_3$, $f_1$; $f_1$, $f_2$ respectively:

$$\cos\omega_1 = h_2h_3g_1,$$
$$\cos\omega_2 = h_3h_1g_2,$$
$$\cos\omega_3 = h_1h_2g_3.$$

**5.05** Orthogonal Curvilinear Coördinates.

$$g_1 = g_2 = g_3 = 0,$$
$$ds^2 = \frac{du^2}{h_1^2} + \frac{dv^2}{h_2^2} + \frac{dw^2}{h_3^2},$$
$$dS_u = \frac{dv\,dw}{h_2h_3},\quad dS_v = \frac{dw\,du}{h_3h_1},\quad dS_w = \frac{du\,dv}{h_1h_2},$$
$$d\tau = \frac{du\,dv\,dw}{h_1h_2h_3}.$$

**5.06** $h_1^2$, $h_2^2$, $h_3^2$ are given by **5.00** (3) and also by:

$$h_1^2 = \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 + \left(\frac{\partial f_1}{\partial z}\right)^2,$$
$$h_2^2 = \left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2 + \left(\frac{\partial f_2}{\partial z}\right)^2,$$
$$h_3^2 = \left(\frac{\partial f_3}{\partial x}\right)^2 + \left(\frac{\partial f_3}{\partial y}\right)^2 + \left(\frac{\partial f_3}{\partial z}\right)^2.$$

**5.07** A vector, **A**, will have three components in the directions of the normals to the orthogonal surfaces $u$, $v$, $w$:

$$A = \sqrt{A_u^2 + A_v^2 + A_w^2}.$$

**5.08**

1. $\operatorname{div} \mathbf{A} = h_1 h_2 h_3 \left\{ \frac{\partial}{\partial u}\left(\frac{A_u}{h_2 h_3}\right) + \frac{\partial}{\partial v}\left(\frac{A_v}{h_3 h_1}\right) + \frac{\partial}{\partial w}\left(\frac{A_w}{h_1 h_2}\right) \right\}.$

2. $\overline{\nabla}^2 = h_1 h_2 h_3 \left\{ \frac{\partial}{\partial u}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_2}{h_3 h_1}\frac{\partial}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_3}{h_1 h_2}\frac{\partial}{\partial w}\right) \right\}.$

3. $$\begin{cases} \operatorname{curl}_u \mathbf{A} = h_2 h_3 \left\{ \frac{\partial}{\partial v}\left(\frac{A_w}{h_3}\right) - \frac{\partial}{\partial w}\left(\frac{A_v}{h_2}\right) \right\}, \\ \operatorname{curl}_v \mathbf{A} = h_3 h_1 \left\{ \frac{\partial}{\partial w}\left(\frac{A_u}{h_1}\right) - \frac{\partial}{\partial u}\left(\frac{A_w}{h_3}\right) \right\}, \\ \operatorname{curl}_w \mathbf{A} = h_1 h_2 \left\{ \frac{\partial}{\partial u}\left(\frac{A_v}{h_2}\right) - \frac{\partial}{\partial v}\left(\frac{A_u}{h_1}\right) \right\}. \end{cases}$$

**5.09** The gradient of a scalar function, $\psi$, has three components in the directions of the normals to the three orthogonal surfaces:

$$h_1 \frac{\partial \psi}{\partial u}, h_2 \frac{\partial \psi}{\partial v}, h_3 \frac{\partial \psi}{\partial w}.$$

**5.20** Spherical Polar Coördinates.

1. $$\begin{cases} u = r, \\ v = \theta, \\ w = \phi. \end{cases}$$

2. $$\begin{cases} x = r \sin\theta \cos\phi. \\ y = r \sin\theta \sin\phi, \\ z = r \cos\theta. \end{cases}$$

3. $$h_1 = 1, \; h_2 = \frac{1}{r}, \; h_3 = \frac{1}{r \sin\theta}.$$

4. $$\begin{cases} dS_r = r^2 \sin\theta \, d\theta \, d\phi, \\ dS_\theta = r \sin\theta \, dr \, d\phi, \\ dS_\phi = r \, dr \, d\theta. \end{cases}$$

5. $$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi.$$

6. $$\operatorname{div} \mathbf{A} = \frac{1}{r^2 \sin\theta} \left\{ \sin\theta \frac{\partial}{\partial r}\left(r^2 A_r\right) + r \frac{\partial}{\partial \theta}\left(\sin\theta \, A_\theta\right) + r \frac{\partial A_\phi}{\partial \phi} \right\}.$$

7. $$\overline{\nabla}^2 = \frac{1}{r^2 \sin\theta} \left\{ \sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial \phi^2} \right\}.$$

8.
$$\begin{cases} \operatorname{curl}_r \mathbf{A} = \dfrac{1}{r\sin\theta}\left\{\dfrac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right) - \dfrac{\partial A_\phi}{\partial\phi}\right\}, \\ \operatorname{curl}_\theta \mathbf{A} = \dfrac{1}{r\sin\theta}\left\{\dfrac{\partial A_r}{\partial\phi} - \sin\theta\dfrac{\partial(rA_\phi)}{\partial r}\right\}, \\ \operatorname{curl}_\phi \mathbf{A} = \dfrac{1}{r}\left\{\dfrac{\partial}{\partial r}\left(r A_\theta\right) - \dfrac{\partial A_r}{\partial\theta}\right\}. \end{cases}$$

## 5.21 Cylindrical Coördinates.

1.
$$\begin{cases} u = \rho, \\ v = \theta, \\ w = z. \end{cases}$$

2.
$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z. \end{cases}$$

3.
$$h_1 = 1, \quad h_2 = \frac{1}{\rho}, \quad h_3 = 1.$$

4.
$$\begin{cases} dS_r = \rho\, d\theta\, dz, \\ dS_\theta = dz\, d\rho, \\ dS_z = \rho\, d\rho\, d\theta. \end{cases}$$

5.
$$d\tau = \rho d\ \rho d\ \theta dz.$$

6.
$$\operatorname{div}\mathbf{A} = \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}\left(\rho A_\rho\right) + \frac{\partial A_\theta}{\partial\theta} + \rho\frac{\partial A_z}{\partial z}\right\}.$$

7.
$$\overline{\nabla}^2 = \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2}{\partial\theta^2} + \rho\frac{\partial^2}{\partial z^2}\right\}.$$

8.
$$\begin{cases} \operatorname{curl}_\rho \mathbf{A} = \dfrac{1}{\rho}\dfrac{\partial A_z}{\partial\theta} - \dfrac{\partial A_\theta}{\partial z}. \\ \operatorname{curl}_\theta \mathbf{A} = \dfrac{\partial A_\rho}{\partial z} - \dfrac{\partial A_z}{\partial\rho}. \\ \operatorname{curl}_z \mathbf{A} = \dfrac{1}{\rho}\left\{\dfrac{\partial}{\partial\rho}(\rho A_\theta) - \dfrac{\partial A_\rho}{\partial\theta}\right\}. \end{cases}$$

## 5.22 Ellipsoidal Coördinates.

$u, v, w$ are the three roots of the equation:

1.
$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1.$$

$a>b>c, \qquad u>v>w.$

$\theta = u$: Ellipsoid.
$\theta = v$: Hyperboloid of one sheet.
$\theta = w$: Hyperboloid of two sheets.

2.
$$\begin{cases} x^2 = \dfrac{(a^2+u)\,(a^2+v)\,(a^2+w)}{(a^2-b^2)\,(a^2-c^2)}, \\ y^2 = -\dfrac{(b^2+u)\,(b^2+v)\,(b^2+w)}{(b^2-c^2)\,(a^2-b^2)}, \\ z^2 = \dfrac{(c^2+u)\,(c^2+v)\,(c^2+w)}{(a^2-c^2)\,(b^2-c^2)}. \end{cases}$$

3.
$$\begin{cases} h_1^2 = \dfrac{4(a^2+u)\,(b^2+u)\,(c^2+u)}{(u-v)\,(u-w)}, \\ h_2^2 = \dfrac{4(a^2+v)\,(b^2+v)\,(c^2+v)}{(v-w)\,(v-u)}, \\ h_3^2 = \dfrac{4(a^2+w)\,(b^2+w)\,(c^2+w)}{(w-u)\,(w-v)}. \end{cases}$$

4. $$\operatorname{div}\mathbf{A} = 2\frac{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}{(u-v)\,(u-w)}\frac{\partial}{\partial u}\left(\sqrt{(u-v)\,(u-w)}\,A_u\right)$$
$$+ 2\frac{\sqrt{(a^2+v)\,(b^2+v)\,(c^2+v)}}{(v-w)\,(u-v)}\frac{\partial}{\partial v}\left(\sqrt{(w-v)\,(u-v)}\,A_v\right)$$
$$+ 2\frac{\sqrt{(a^2+w)\,(b^2+w)\,(c^2+w)}}{(u-w)\,(v-w)}\frac{\partial}{\partial w}\left(\sqrt{(u-w)\,(v-w)}\,A_w\right).$$

5. $$\overline{\nabla}^2 = 4\frac{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}{(u-v)\,(u-w)}\frac{\partial}{\partial u}\left(\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}\frac{\partial}{\partial u}\right)$$
$$+ 4\frac{\sqrt{(a^2+v)\,(b^2+v)\,(c^2+v)}}{(u-v)\,(v-w)}\frac{\partial}{\partial v}\left(\sqrt{(a^2+v)\,(b^2+v)\,(c^2+v)}\frac{\partial}{\partial v}\right)$$
$$+ 4\frac{\sqrt{(a^2+w)\,(b^2+w)\,(c^2+w)}}{(a-w)\,(v-w)}\frac{\partial}{\partial w}\left(\sqrt{(a^2+w)\,(b^2+w)\,(c^2+w)}\frac{\partial}{\delta w}\right).$$

6.
$$\begin{cases} \operatorname{curl}_u\mathbf{A} = \dfrac{2}{v-w}\left\{\sqrt{\dfrac{(a^2+v)\,(b^2+v)\,(c^2+v)}{u-v}}\dfrac{\partial}{\partial v}\left(\sqrt{w-v}\,A_w\right)\right. \\ \qquad\qquad \left. -\sqrt{\dfrac{(a^2+w)\,(b^2+w)\,(c^2+w)}{u-w}}\dfrac{\partial}{\partial w}\left(\sqrt{v-w}\,A_v\right)\right\}. \end{cases}$$

7.
$$\begin{cases} \operatorname{curl}_v\mathbf{A} = \dfrac{2}{u-w}\left\{\sqrt{\dfrac{(a^2+w)\,(b^2+w)\,(c^2+w)}{v-w}}\dfrac{\partial}{\partial w}\left(\sqrt{u-w}\,A_u\right)\right. \\ \qquad\qquad \left. -\sqrt{\dfrac{(a^2+u)\,(b^2+u)\,(c^2+u)}{v-u}}\dfrac{\partial}{\partial u}\left(\sqrt{w-u}\,A_w\right)\right\} \\ \operatorname{curl}_w\mathbf{A} = \dfrac{2}{u-v}\left\{\sqrt{\dfrac{(a^2+u)\,(b^2+u)\,(c^2+u)}{w-u}}\dfrac{\partial}{\partial u}\left(\sqrt{v-u}\,A_v\right)\right. \\ \qquad\qquad \left. -\sqrt{\dfrac{(a^2+v)\,(b^2+v)\,(c^2+v)}{w-v}}\dfrac{\partial}{\partial v}\left(\sqrt{u-v}\,A_u\right)\right\}. \end{cases}$$

**5.23** Conical Coördinates.

The three orthogonal surfaces are: the spheres,

1. $$x^2 + y^2 + z^2 = u^2,$$

the two cones:

2. $$\frac{x^2}{v^2} + \frac{y^2}{v^2 - b^2} + \frac{z^2}{v^2 - c^2} = 0.$$

3. $$\frac{x^2}{w^2} + \frac{y^2}{w^2 - b^2} + \frac{z^2}{w^2 - c^2} = 0.$$

$$c^2 > v^2 > b^2 > w^2.$$

4. $$\begin{cases} x^2 = \dfrac{u^2 v^2 w^2}{b^2 c^2}, \\ y^2 = \dfrac{u^2 (v^2 - b^2)\,(w^2 - b^2)}{b^2 (b^2 - c^2)}, \\ z^2 = \dfrac{u^2 (v^2 - c^2)\,(w^2 - c^2)}{c^2 (c^2 - b^2)}. \end{cases}$$

5. $$h_1 = 1, \quad h_2^2 = \frac{(v^2 - b^2)\,(c^2 - v^2)}{u^2 (v^2 - w^2)}, \quad h_3^2 = \frac{(b^2 - w^2)\,(c^2 - w^2)}{u^2 (v^2 - w^2)}.$$

6. $$\operatorname{div} \mathbf{A} = \frac{1}{u^2} \frac{\partial}{\partial u} (u^2 A_u) + \frac{\sqrt{(v^2 - b^2)\,(c^2 - v^2)}}{u (v^2 - w^2)} \frac{\partial}{\partial v} \left( \sqrt{v^2 - w^2}\, A_v \right) + \frac{\sqrt{(b^2 - w^2)\,(c^2 - w^2)}}{u (v^2 - w^2)} \frac{\partial}{\partial w} \left( \sqrt{v^2 - w^2}\, A_w \right).$$

7. $$\nabla^2 = \frac{1}{u^2} \frac{\partial}{\partial u} \left( u^2 \frac{\partial}{\partial u} \right) + \frac{\sqrt{(v^2 - b^2)\,(c^2 - v^2)}}{u^2 (v^2 - w^2)} \frac{\partial}{\partial v} \left( \sqrt{(v^2 - b^2)\,(c^2 - v^2)} \frac{\partial}{\partial v} \right) + \frac{\sqrt{(b^2 - w^2)\,(c^2 - w^2)}}{u^2 (v^2 - w^2)} \frac{\partial}{\partial w} \left( \sqrt{(b^2 - w^2)\,(c^2 - w^2)} \frac{\partial}{\partial w} \right).$$

8. $$\begin{cases} \operatorname{curl}_u \mathbf{A} = \dfrac{1}{u (v^2 - w^2)} \left\{ \sqrt{(v^2 - b^2)\,(c^2 - v^2)} \dfrac{\partial}{\partial v} \left( \sqrt{v^2 - w^2}\, A_w \right) - \sqrt{(b^2 - w^2)\,(c^2 - w^2)} \dfrac{\partial}{\partial w} \left( \sqrt{v^2 - w^2}\, A_v \right) \right\}, \\ \operatorname{curl}_v \mathbf{A} = \dfrac{\sqrt{(b^2 - w^2)\,(c^2 - w^2)}}{u \sqrt{v^2 - w^2}} \dfrac{\partial A_u}{\partial w} - \dfrac{1}{u} \dfrac{\partial}{\partial u} \left( u A_u \right) \Big\}, \\ \operatorname{curl}_w \mathbf{A} = \dfrac{1}{u} \dfrac{\partial}{\partial u} \left( u A_v \right) - \dfrac{\sqrt{(v^2 - b^2)\,(c^2 - v^2)}}{u \sqrt{v^2 - w^2}} \dfrac{\partial A_u}{\partial v}. \end{cases}$$

**5.30** Elliptic Cylinder Coördinates.

The three orthogonal surfaces are:

1. The elliptic cylinders:

$$\frac{x^2}{c^2 u^2} + \frac{y^2}{c^2 (u^2 - 1)} = 1.$$

2. The hyperbolic cylinders:

$$\frac{x^2}{c^2v^2} - \frac{y^2}{c^2(1 - v^2)} = 1.$$

3. The planes: $z = w.$

$2c$ is the distance between the foci of the confocal ellipses and hyperbolas:

4. $x = cuv.$

5. $y = c\sqrt{u^2 - 1}\ \sqrt{1 - v^2}.$

6. $$\frac{1}{h_1^2} = \frac{1}{h_2^2} = c^2(u^2 - v^2), \quad h_3 = 1.$$

7. $$\operatorname{div} \mathbf{A} = \frac{1}{c(u^2 - v^2)} \left\{ \frac{\partial}{\partial u}\left(\sqrt{u^2 - v^2}\, A_u\right) + \frac{\partial}{\partial v}\left(\sqrt{u^2 - v^2}\, A_v\right) \right\} + \frac{\partial A_z}{\partial z}.$$

8. $$\overline{\nabla}^2 = \frac{1}{c^2(u^2 - v^2)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{\partial^2}{\partial z^2}.$$

9. $$\begin{cases} \operatorname{curl}_u \mathbf{A} = \dfrac{1}{c\sqrt{u^2 - v^2}} \dfrac{\partial A_z}{\partial v} - \dfrac{\partial A_v}{\partial z}, \\ \operatorname{curl}_v \mathbf{A} = \dfrac{\partial A_u}{\partial z} - \dfrac{1}{c\sqrt{u^2 - v^2}} \dfrac{\partial A_z}{\partial u}, \\ \operatorname{curl}_z \mathbf{A} = \dfrac{1}{c(u^2 - v^2)} \left\{ \dfrac{\partial}{\partial u}\left(\sqrt{u^2 - v^2}\, A_v\right) - \dfrac{\partial}{\partial v}\left(\sqrt{u^2 - v^2}\, A_u\right) \right\}. \end{cases}$$

**5.31** Parabolic Cylinder Coördinates.

The three orthogonal surfaces are the two parabolic cylinders:

1. $y^2 = 4cux + 4c^2u^2.$

2. $y^2 = -4cvx + 4c^2v^2.$

And the planes:

3. $z = w.$

4. $x = c(v - u).$

5. $y = 2c\sqrt{uv}.$

6. $$\frac{1}{h_1^2} = \frac{u + v}{u}, \quad \frac{1}{h_2^2} = \frac{u + v}{v}, \quad h_3 = 1.$$

7. $$\operatorname{div} \mathbf{A} = \frac{\sqrt{uv}}{u + v} \left\{ \frac{\partial}{\partial u}\left(\sqrt{\frac{u + v}{v}}\, A_u\right) + \frac{\partial}{\partial v}\left(\sqrt{\frac{u + v}{u}}\, A_v\right) \right\} + \frac{\partial A_z}{\partial z}.$$

8. $$\overline{\nabla}^2 = \frac{\sqrt{uv}}{u + v} \left\{ \frac{\partial}{\partial u}\left(\frac{u}{v}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{v}{u}\frac{\partial}{\partial v}\right) \right\} + \frac{\partial^2}{\partial z^2}.$$

$$9.\quad \begin{cases} \operatorname{curl}_u \mathbf{A} = \sqrt{\dfrac{v}{u+v}}\ \dfrac{\partial A_z}{\partial v} - \dfrac{v}{u+v}\ \dfrac{\partial A_v}{\partial z}, \\ \operatorname{curl}_v \mathbf{A} = \dfrac{u}{u+v}\ \dfrac{\partial A_u}{\partial z} - \sqrt{\dfrac{u}{u+v}}\ \dfrac{\partial A_z}{\partial u}, \\ \operatorname{curl}_z \mathbf{A} = \dfrac{\sqrt{uv}}{u+v}\left\{ \dfrac{\partial}{\partial u}\left(\sqrt{\dfrac{v}{u+v}}\, A_v\right) - \dfrac{\partial}{\partial v}\left(\sqrt{\dfrac{u}{u+v}}\, A_u\right)\right\}. \end{cases}$$

**5.40** Helical Coördinates. (Nicholson, Phil. Mag. 19, 77, 1910.)

A cylinder of any cross-section is wound on a circular cylinder in the form of a helix of angle $\alpha$. $a$ = radius of circular cylinder on which the central line of the normal cross-sections of the helical cylinder lies. The $z$-axis is along the axis of the cylinder of radius $a$.

$u = \rho$ and $v = \phi$ are the polar coördinates in the plane of any normal section of the helical cylinder. $\phi$ is measured from a line perpendicular to $z$ and to the tangent to the cylinder.

$w = \theta$ = the twist in a plane perpendicular to $z$ of the radius in that plane measured from a line parallel to the $x$-axis:

$$1.\quad \begin{cases} x = (a + \rho\cos\phi)\cos\theta + \rho\sin\alpha\sin\theta\sin\phi, \\ y = (a + \rho\cos\phi)\sin\theta - \rho\sin\alpha\cos\theta\sin\phi, \\ z = a\,\theta\tan\alpha + \rho\cos\alpha\sin\phi. \end{cases}$$

$$2.\quad \begin{cases} h_1 = 1, \quad h_2 = \dfrac{1}{\rho}, \\ h_3^2 = \dfrac{1}{a^2\sec^2\alpha + 2a\rho\cos\phi + \rho^2(\cos^2\phi + \sin^2\alpha\sin^2\phi)}. \end{cases}$$

**5.50** Surfaces of Revolution.

$z$-axis = axis of revolution.

$\rho$, $\theta$ = polar coördinates in any plane perpendicular to $z$-axis.

$$1.\quad ds^2 = dz^2 + d\rho^2 + \rho^2 d\theta^2 = \frac{du^2}{h_1^2} + \frac{dv^2}{h_2^2} + \frac{dw^2}{h_3^2}.$$

In any meridian plane, $z$, $\rho$, determine $u$, $v$, from:

$$2.\quad f(z + i\rho) = u + iv.$$

$$3.\quad w = \theta.$$

Then $u$, $v$, $\theta$ will form a system of orthogonal curvilinear coördinates.

**5.51** Spheroidal Coördinates (Prolate Spheroids):

1. $$z + i\rho = c \cosh (u + iv).$$

2. $$\begin{cases} z = c \cosh u \cos v, \\ \rho = c \sinh u \sin v. \end{cases}$$

The three orthogonal surfaces are the ellipsoids and hyperboloids of revolution, and the planes, $\theta$:

3. $$\begin{cases} \dfrac{z^2}{c^2 \cosh^2 u} + \dfrac{\rho^2}{c^2 \sinh^2 u} = 1, \\ \dfrac{z^2}{c^2 \cos^2 v} - \dfrac{\rho^2}{c^2 \sin^2 v} = 1. \end{cases}$$

With $\cos u = \lambda$, $\cos v = \mu$:

4. $$\begin{cases} z = c\,\lambda\,\mu, \\ \rho = c\sqrt{(\lambda^2 - 1)\,(1 - \mu^2)}. \end{cases}$$

5. $$h_1^2 = \frac{\lambda^2 - 1}{c^2(\lambda^2 - \mu^2)}, \quad h_2^2 = \frac{1 - \mu^2}{c^2\,(\lambda^2 - \mu^2)}, \quad h_3^2 = \frac{1}{c^2(\lambda^2 - 1)\,(1 - \mu^2)}.$$

**5.52** Spheroidal Coördinates (Oblate Spheroids):

1. $$\rho + iz = c \cosh(u + iv).$$

2. $$\begin{aligned} z &= c \sinh u \sin v. \\ \rho &= c \cosh u \cos v. \end{aligned}$$

3. $$\cosh u = \lambda, \quad \cos v = \mu.$$

4. $$h_1^2 = \frac{1 - \mu^2}{c^2(\lambda^2 - \mu^2)}, \quad h_2^2 = \frac{\lambda^2 - 1}{c^2(\lambda^2 - \mu^2)}, \quad h_3^2 = \frac{1}{c^2(\lambda^2 - 1)\,(1 - \mu^2)}.$$

**5.53** Parabolic Coördinates:

1. $$z + i\rho = c(u + iv)^2.$$

2. $$\begin{cases} z = c(u^2 - v^2), \\ \rho = 2cuv. \end{cases}$$

3. $$u^2 = \lambda, \quad v^2 = \mu.$$

With curvilinear coördinates, $\lambda$, $\mu$, $\theta$:

4. $$h_1 = \frac{1}{c}\sqrt{\frac{\lambda}{\lambda+\mu}}, \quad h_2 = \frac{1}{c}\sqrt{\frac{\mu}{\lambda+\mu}}, \quad h_3 = \frac{1}{2c\sqrt{\lambda\mu}}.$$

**5.54** Toroidal Coördinates:

1. $$u + iv = \log \frac{z + a + i\rho}{z - a + i\rho},$$

$$\rho = \frac{a \sinh u}{\cosh u - \cos v}.$$

2. $$z = \frac{a \sin v}{\cosh u - \cos v}.$$

3. $$h_1 = h_2 = \frac{\cosh u - \cos v}{a}, \quad h_3 = \frac{\cosh u - \cos v}{a \sinh u}.$$

The three orthogonal surfaces are:

(a) Anchor rings, whose axial circles have radii,

$$a \coth u,$$

and whose cross-sections are circles of radii,

$$a \operatorname{csch} u;$$

(b) Spheres, whose centers are on the axis of revolution at distances,

$$\pm a \cot v,$$

from the origin, whose radii are,

$$a \csc v,$$

and which accordingly have a common circle,

$$\rho = a, \ z = 0;$$

(c) Planes through the axis,

$$w = \theta = \text{const.}$$

# VI. INFINITE SERIES

**6.00** An infinite series:

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots .$$

is absolutely convergent if the series formed of the moduli of its terms:

$$| u_1 | + | u_2 | + | u_2 | + \cdots .$$

is convergent.

A series which is convergent, but whose moduli do not form a convergent series, is conditionally convergent.

TESTS FOR CONVERGENCE

**6.011** Comparison test. The series $\Sigma u_n$ is absolutely convergent if $| u_n |$ is less than $C | v_n |$ where $C$ is a number independent of $n$, and $v_n$ is the $n$th term of another series which is known to be absolutely convergent.

**6.012** Cauchy's test. If

$$\lim_{n \to \infty} | u_n |^{\frac{1}{n}} < 1,$$

the series $\Sigma u_n$ is absolutely convergent.

**6.013** D'Alembert's test. If for all values of $n$ greater than some fixed value, $r$, the ratio $\left| \frac{u_{n+1}}{u_n} \right|$ is less than $\rho$, where $\rho$ is a positive number less than unity and independent of $n$, the series $\Sigma u_n$ is absolutely convergent.

**6.014** Cauchy's integral test. Let $f(x)$ be a steadily decreasing positive function such that,

$$f(n) \geqslant a_n.$$

Then the positive term series $\Sigma a_n$ is convergent if,

$$\int_m^{\infty} f(x)dx,$$

is convergent.

**6.015** Raabe's test. The positive term series $\Sigma a_n$ is convergent if,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geqslant l \quad \text{where } l > 1.$$

It is divergent if,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \leqslant 1 .$$

**6.020** Alternating series. A series of real terms, alternately positive and negative, is convergent if $a_{n+1} \leqslant a_n$ and

$$\lim_{n \to \infty} a_n = 0.$$

In such a series the sum of the first $s$ terms differs from the sum of the series by a quantity less than the numerical value of the $(s+1)st$ term.

**6.025** If $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1$, the series $\Sigma u_n$ will be absolutely convergent if there is a positive number $c$, independent of $n$, such that,

$$\lim_{n \to \infty} n \left\{ \left| \frac{u_{n+1}}{u_n} \right| - 1 \right\} = -1 - c.$$

**6.030** The sum of an absolutely convergent series is not affected by changing the order in which the terms occur.

**6.031** Two absolutely convergent series,

$$S = u_1 + u_2 + u_3 + \ldots\ldots$$
$$T = v_1 + v_2 + v_3 + \ldots\ldots$$

may be multiplied together, and the sum of the products of their terms, written in any order, is $ST$,

$$ST = u_1v_1 + u_2v_1 + u_1v_2 + \ldots\ldots$$

**6.032** An absolutely convergent power series may be differentiated or integrated term by term and the resulting series will be absolutely convergent and equal to the differential or integral of the sum of the given series.

**6.040** Uniform Convergence. An infinite series of functions of $x$,

$$S(x) = u_1(x) + u_2(x) + u_3(x) + \ldots\ldots$$

is uniformly convergent within a certain region of the variable $x$ if a finite number, $N$, can be found such that for all values of $n \geqslant N$ the absolute value of the remainder, $R_n$ | after $n$ terms is less than an assigned arbitrary small quantity $\epsilon$ at all points within the given range.

Example. The series,

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n},$$

is absolutely convergent for all real values of $x$. Its sum is $1 + x^2$ if $x$ is not zero. If $x$ is zero the sum is zero. The series is non-uniformly convergent in the neighborhood of $x = 0$.

**6.041** A uniformly convergent series is not necessarily absolutely convergent, nor is an absolutely convergent series necessarily uniformly convergent.

**6.042** A sufficient, though not necessary, test for uniform convergence is as follows:

If for all values of $x$ within a certain region the moduli of the terms of the series,

$$S = u_1(x) + u_2(x) + \ldots\ldots$$

are less than the corresponding terms of a convergent series of positive terms,

$$T = M_1 + M_2 + M_3 + \ldots\ldots$$

where $M_n$ is independent of $x$, then the series $S$ is uniformly convergent in the given region.

**6.043** A power series is uniformly convergent at all points within its circle of convergence.

**6.044** A uniformly convergent series,

$$S = u_1(x) + u_2(x) + \ldots\ldots$$

may be integrated term by term, and,

$$\int S\,dx = \sum_{n=1}^{\infty} \int u_n(x)\,dx.$$

**6.045** A uniformly convergent series,

$$S = u_1(x) + u_2(x) + \ldots\ldots$$

may be differentiated term by term, and if the resulting series is uniformly convergent,

$$\frac{d}{dx} S = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x).$$

**6.100** Taylor's theorem.

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots + \frac{h^n}{n!} f^{(n)}(x) + R_n.$$

**6.101** Lagrange's form for the remainder:

$$R_n = f^{(n+1)}(x + \theta h) \cdot \frac{h^{n+1}}{(n+1)!};\quad 0 < \theta < 1.$$

**6.102** Cauchy's form for the remainder:

$$R_n = f^{(n+1)}(x + \theta h)\,\frac{h^{n+1}(1-\theta)^n}{n!};\quad 0 < \theta < 1.$$

**6.103**

$$f(x) = f(h) + f'(h)\cdot\frac{x-h}{1!} + f''(h)\cdot\frac{(x-h)^2}{2!} + \ldots + f^{(n)}(h)\,\frac{(x-h)^n}{n!} + R_n$$

$$R_n = f^{(n+1)}\{h + \theta\,(x-h)\}\,\frac{(x-h)^{n+1}}{(n+1)!} \qquad 0<\theta<1.$$

**6.104** Maclaurin's theorem:

$$f(x) = f(0) + f'(0)\,\frac{x}{1!} + f''(0)\,\frac{x^2}{2!} + \ldots + f^{(n)}(0)\,\frac{x^n}{n!} + R_n$$

$$R_n = f^{(n+1)}(\theta x)\,\frac{x^{n+1}}{(n+1)!}\,(1-\theta)^n; \qquad 0<\theta<1.$$

**6.105** Lagrange's theorem. Given:

$$y = z + x\phi(y).$$

The expansion of $f(y)$ in powers of $x$ is:

$$f(y) = f(z) + x\phi(z)f'(z) + \frac{x^2}{2!}\,\frac{d}{dz}\,[\{\phi(z)\}^2 f'(z)] + \ldots + \frac{x^n}{n!}\,\frac{d^{n-1}}{dz^{n-1}}\,[\{\phi(z)\}^n f'(z)] + \ldots$$

## SYMBOLIC REPRESENTATION OF INFINITE SERIES

**6.150** The infinite series:

$$f(x) = 1 + a_1x + \frac{1}{2!}\,a_2x^2 + \frac{1}{3!}\,a_3x^3 + \ldots + \frac{1}{k!}\,a_kx^k + \ldots$$

may be written:

$$f(x) = e^{ax},$$

where $a^k$ is interpreted as equivalent to $a_k$.

**6.151** The infinite series, written without factorials,

$$f(x) = 1 + a_1x + a_2x^2 + \ldots + a_kx^k + \ldots$$

may be written:

$$f(x) = \frac{1}{1-ax},$$

where $a^k$ is interpreted as equivalent to $a_k$.

**6.152** Symbolic form of Taylor's theorem:

$$f(x+h) = e^{h\frac{\partial}{\partial x}} f(x).$$

**6.153** Taylor's theorem for functions of many variables:

$$f(x_1 + h_1,\, x_2 + h_2,\, \ldots) = e^{h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \cdots} f(x_1, x_2, \ldots)$$

$$= f(x_1, x_2, \ldots) + h_1\frac{\partial f}{\partial x_1} + h_2\frac{\partial f}{\partial x_2} + \ldots$$

$$+ \frac{h_1^2}{2!}\,\frac{\partial^2 f}{\partial x_1^2} + \frac{2}{2!}\,h_1h_2\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{h_2^2}{2!}\,\frac{\partial^2 f}{\partial x_2^2} + \ldots$$

$$+ \ldots$$

## TRANSFORMATION OF INFINITE SERIES

Series which converge slowly may often be transformed to more rapidly converging series by the following methods.

**6.20** Euler's transformation formula:

$$S = a_0 + a_1 x + a_2 x^2 + \cdots\cdots\cdots$$

$$= \frac{1}{1-x} a_0 + \frac{1}{1-x} \sum_{k=1}^{\infty} \left(\frac{x}{1-x}\right)^k \Delta^k a_0,$$

where:

$$\Delta a_0 = a_1 - a_0,$$

$$\Delta^2 a_0 = \Delta a_1 - \Delta a_0 = a_2 - 2a_1 + a_0,$$

$$\Delta^3 a_0 = \Delta^2 a_1 - \Delta^2 a_0 = a_3 - 3a_2 + 3a_1 - a_0,$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\Delta^k a_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} a_{k+n-m}.$$

The second series may converge more rapidly than the first.

Example 1.

$$S = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1},$$

$$x = -1, \quad a_k = \frac{1}{2k+1},$$

$$S = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k+1)}.$$

Example 2.

$$S = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} = \log 2,$$

$$x = -1, \quad a_k = \frac{1}{k+1}.$$

$$S = \sum_{k=1}^{\infty} \frac{1}{k 2^k},$$

**6.21** Markoff's transformation formula. (Differenzenrechnung, p. 180.)

$$\sum_{k=0}^{n} a_k x^k - \left(\frac{x}{1-x}\right)^m \sum_{k=0}^{n} x^k \Delta^m a_k = \sum_{k=0}^{m} \frac{x^k}{(1-x)^{k+1}} \Delta^k a_0 - \sum_{k=0}^{m} \frac{x^{k+n}}{(1-x)^{k+1}} \Delta^k a_n.$$

**6.22** Kummer's transformation.

$A_0, A_1, A_2, \ldots$ is a sequence of positive numbers such that

$$\lambda_m = A_m - A_{m+1}\frac{a_{m+1}}{a_m},$$

and

$$\operatorname*{Limit}_{m\to\infty} \lambda_m,$$

approaches a definite positive value. Usually this limit can be taken as unity. If not, it is only necessary to divide $A_m$ by this limit:

$$\alpha = \operatorname*{Limit}_{m\to\infty} A_m a_m.$$

Then:

$$\sum_{m=n}^{\infty} a_m = (A_n a_n - \alpha) + \sum_{m=n}^{\infty} (1 - \lambda_m) a_m.$$

Example 1.

$$S = \sum_{m=1}^{\infty} \frac{1}{m^2},$$

$$A_m = m, \quad \lambda_m = \frac{m}{m+1}, \quad \operatorname*{Limit}_{m\to\infty} \lambda_m = 1,$$

$$\alpha = 0$$

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = 1 + \sum_{m=1}^{\infty} \frac{1}{(m+1)m^2}.$$

Applying the transformation to the series on the right:

$$A_m = \frac{m}{2}, \quad \lambda_m = \frac{m}{m+2}, \quad \alpha = 0,$$

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{2^2} + 2\sum_{m=1}^{\infty} \frac{1}{m^2(m+1)(m+2)}.$$

Applying the transformation $n$ times:

$$\sum_{m=n+1}^{\infty} \frac{1}{m^2} = n!\sum_{m=1}^{\infty} \frac{1}{m^2(m+1)(m+2)\ldots(m+n)}.$$

Example 2.

$$S = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{2m-1},$$

$$A_m = \frac{1}{2}, \quad \lambda_m = \frac{2m}{2m+1}, \quad \alpha = 0,$$

$$S = \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{4m^2-1}.$$

Applying the transformation again, with:

$$A_m = \frac{1}{2}\frac{2m+1}{2m-1}, \quad \lambda_m = \frac{4m^2+1}{4m^2-1}, \quad \alpha = 0,$$

$$S = 1 - 2\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(4m^2-1)^2}.$$

Applying the transformation again, with:

$$A_m = \frac{1}{2}\frac{2m+1}{2m-3}, \quad \lambda_m = \frac{4m^2+3}{4m^2-9}, \quad \alpha = 0,$$

$$S = \frac{4}{3} + 24\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(4m^2-1)^2(4m^2-9)}.$$

Example 3.

$$S = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(2m-1)^2},$$

$$A_m = \frac{2m-1}{2(2m-3)}, \quad \lambda_m = \frac{4m^2-4m+1}{(2m-3)(2m+1)}, \quad \alpha = 0,$$

$$S = \frac{5}{6} + 4\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(2m-1)(2m+3)(2m+1)^2}.$$

**6.23** Leclert's modification of Kummer's transformation. With the same notation as in **6.22** and,

$$\lim_{m\to\infty}\lambda_m = \omega,$$

$$\sum_{n=0}^{\infty}a_n = a_0 + \frac{A_1a_1}{\lambda_1} - \frac{\alpha}{\omega} + \sum_{m=1}^{\infty}\left(\frac{1}{\lambda_{m+1}} - \frac{1}{\lambda_m}\right)A_{m+1}\,a_{m+1}.$$

Example 1.

$$S = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1},$$

$$a_0 = 0, \quad A_m = 1, \quad \omega = 2, \quad \alpha = 0, \quad \lambda_m = \frac{4m}{2m+1},$$

$$S = \frac{3}{4} + \frac{1}{4}\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m(2m+1)(m+1)}.$$

Applying the transformation to the series on the right, with:

$$a_0 = 0, \quad A_m = \frac{2m+1}{m-1}, \quad \lambda_m = \frac{(2m+1)^2}{(m-1)(m+2)}, \quad \omega = 4, \quad \alpha = 0,$$

$$S = \frac{19}{24} + \frac{9}{2}\sum_{m=1}^{\infty}(-1)^m \frac{1}{m(m+2)(2m+1)^2(2m+3)^2}.$$

**6.26** Reversion of series. The power series:

$$z = x - b_1x^2 - b_2x^3 - b_3x^4 - \ldots\ldots$$

may be reversed, yielding:

$$x = z + c_1z^2 + c_2z^3 + c_3z^4 + \ldots\ldots$$

where:

$c_1 = b_1,$

$c_2 = b_2 + 2b_1^2,$

$c_3 = b_3 + 5b_1b_2 + 5b_1^3,$

$c_4 = b_4 + 6b_1b_3 + 3b_2^2 + 21b_1^2b_2 + 14b_1^4,$

$c_5 = b_5 + 7(b_1b_4 + b_2b_3) + 28(b_1^2b_3 + b_1b_2^2) + 84b_1^3b_2 + 42b_1^5,$

$c_6 = b_6 + 4(2b_1b_5 + 2b_2b_4 + b_3^2) + 12(3b_1^2b_4 + 6b_1b_2b_3 + b_2^3)$
$+ 60(2b_1^3b_3 + 3b_1^2b_2^2) + 330b_1^4b_2 + 132b_1^6,$

$c_7 = b_7 + 9(b_1b_6 + b_2b_5 + b_3b_4) + 45(b_1^2b_5 + b_1b_3^2 + b_2^2b_3 + 2b_1b_2b_4)$
$+ 165(b_1^3b_4 + b_1b_2^3 + 3b_1^2b_2b_3) + 495(b_1^4b_3 + 2b_1^3b_2^2)$
$+ 1287b_1^5b_2 + 429b_1^7.$

Van Orstrand (Phil. Mag. 19, 366, 1910) gives the coefficients of the reversed series up to $c_{12}$.

**6.30** Binomial series.

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$$

$$+ \frac{n!}{(n-k)!k!}x^k + \ldots. = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots.\binom{n}{k}x^k + \ldots$$

**6.31** Convergence of the binomial series.

The series converges absolutely for $|x|<1$ and diverges for $|x|>1$. When $x=1$, the series converges for $n>-1$ and diverges for $n\leqslant -1$. It is absolutely convergent only for $n>0$.

When $x=-1$ it is absolutely convergent for $n>0$, and divergent for $n<0$.

**6.32** Special cases of the binomial series.

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = b^n\left(1+\frac{a}{b}\right)^n.$$

If $\left|\frac{b}{a}\right|<1$ put $x=\frac{b}{a}$ in **6.30**; if $\left|\frac{b}{a}\right|>1$ put $x=\frac{a}{b}$ in **6.30.**

**6.33**

1. $(1+x)^{\frac{n}{m}} = 1+\frac{n}{m}x-\frac{n(m-n)}{2!m^2}x^2+\frac{n(m-n)(2m-n)}{3!m^3}x^3-$
$$\ldots\ldots+(-1)^k\frac{n(m-n)(2m-n)\ldots\ldots[(k-1)m-n]}{k!m^k}x^k$$
$$+\ldots.$$

2. $(1+x)^{-1} = 1-x+x^2-x^3+x^4-\ldots\ldots$

3. $(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4-\ldots.$

4. $\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1\cdot 1}{2\cdot 4}x^2+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4+\ldots$

5. $\frac{1}{\sqrt{1+x}} = 1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4-\ldots$

6. $(1+x)^{\frac{1}{3}} = 1+\frac{1}{3}x-\frac{1\cdot 2}{3\cdot 6}x^2+\frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}x^3-\frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}x^4+\ldots.$

7. $(1+x)^{-\frac{1}{3}} = 1-\frac{1}{3}x+\frac{1\cdot 4}{3\cdot 6}x^2-\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3+\frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}x^4-\ldots$

8. $(1+x)^{\frac{3}{2}} = 1+\frac{3}{2}x+\frac{3\cdot 1}{2\cdot 4}x^2-\frac{3\cdot 1\cdot 1}{2\cdot 4\cdot 6}x^3+\frac{3\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8}x^4-\frac{3\cdot 1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^5+\ldots.$

9. $(1+x)^{-\frac{3}{2}} = 1-\frac{3}{2}x+\frac{3\cdot 5}{2\cdot 4}x^2-\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}x^3+\ldots\ldots$

10. $(1+x)^{\frac{1}{4}} = 1+\frac{1}{4}x-\frac{3}{32}x^2+\frac{7}{128}x^3-\frac{77}{2048}x^4+\ldots$

11. $(1+x)^{-\frac{1}{4}} = 1-\frac{1}{4}x+\frac{5}{32}x^2-\frac{15}{128}x^3+\frac{195}{2048}x^4-\ldots$

12. $(1+x)^{\frac{1}{5}} = 1+\frac{1}{5}x-\frac{2}{25}x^2+\frac{6}{125}x^3-\frac{21}{625}x^4+\ldots.$

13. $(1+x)^{-\frac{1}{5}} = 1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \ldots$

14. $(1+x)^{\frac{1}{6}} = 1 + \frac{1}{6}x - \frac{5}{72}x^2 + \frac{55}{1296}x^3 - \frac{935}{31104}x^4 + \ldots$

15. $(1+x)^{-\frac{1}{6}} = 1 - \frac{1}{6}x + \frac{7}{72}x^2 - \frac{91}{1296}x^3 + \frac{1729}{31104}x^4 - \ldots$

**6.350**

1. $\frac{x}{1-x} = \frac{x}{1+x} + \frac{2x^2}{1+x^2} + \frac{4x^4}{1+x^4} + \frac{8x^8}{1+x^8} + \ldots$ $[x^2<1]$.

2. $\frac{x}{1-x} = \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \ldots$ $[x^2<1]$.

3. $\frac{1}{x-1} = \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \ldots$ $[x^2>1]$.

**6.351**

1. $$\left\{1+\sqrt{1+x}\right\}^n = 2^n\left\{1 + n\left(\frac{x}{4}\right) + \frac{n(n-3)}{2!}\left(\frac{x}{4}\right)^2 + \frac{n(n-4)(n-5)}{3!}\left(\frac{x}{4}\right)^3 + \ldots\right\}. \qquad [x^2<1].$$

$n$ may be any real number.

2. $$\left(x+\sqrt{1+x^2}\right)^n = 1 + \frac{n^2}{2!}x^2 + \frac{n^2(n^2-2^2)}{4!}x^4 + \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}x^6 + \ldots + \frac{n}{1!}x + \frac{n(n^2-1^2)}{3!}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 + \ldots \qquad [x^2<1].$$

**6.352** If $a$ is a positive integer:

$$\frac{1}{a} + \frac{1}{a(a+1)}x + \frac{1}{a(a+1)(a+2)}x^2 + \ldots = \frac{(a-1)!}{x^a}\left\{e^x - \sum_{n=0}^{a-1}\frac{x^n}{n!}\right\}.$$

**6.353** If $a$ and $b$ are positive integers, and $a<b$:

$$\frac{a}{b} + \frac{a(a+1)}{b(b+1)}x + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^2 + \ldots = (b-a)\binom{b-1}{a-1}\left\{\frac{(-1)^{b-a}\log(1-x)}{x^b}(1-x)^{b-a-1} + \frac{1}{x^a}\sum_{k=1}^{b-a}(-1)^k\binom{b-a-1}{k-1}\sum_{n=1}^{a+k-1}\frac{x^{n-k}}{n}\right\}.$$

(Schwatt, Phil. Mag. 31, 75, 1916)

## POLYNOMIAL SERIES

**6.360**
$$\frac{b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots}{a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots} = \frac{1}{a_0}(c_0 + c_1x + c_2x^2 + \ldots),$$

$$c_0 - b_0 = 0,$$

$$c_1 + \frac{c_0a_1}{a_0} - b_1 = 0,$$

$$c_2 + \frac{c_1a_1}{a_0} + \frac{c_0a_2}{a_0} - b_2 = 0,$$

$$c_3 + \frac{c_2a_1}{a_0} + \frac{c_1a_2}{a_0} + \frac{c_0a_3}{a_0} - b_3 = 0.$$

. . . . .

. . . . .

$$c_n = \frac{(-1)^n}{{a_0}^n}\begin{vmatrix} (a_1b_0 - a_0b_1) & a_0 & 0 & \ldots\ldots & 0 \\ (a_2b_0 - a_0b_2) & a_1 & a_0 & \ldots\ldots & 0 \\ (a_3b_0 - a_0b_3) & a_2 & a_1 & \ldots\ldots & 0 \\ \ldots\ldots\ldots & & & & \\ \ldots\ldots\ldots & & & & \\ (a_{n-1}b_0 - a_0b_{n-1}) & a_{n-2} & a_{n-3} & \ldots\ldots & a_0 \\ (a_nb_0 - a_0b_n) & a_{n-1} & a_{n-2} & \ldots\ldots & a_1 \end{vmatrix}$$

**6.361**
$$(a_0 + a_1x + a_2x^2 + \ldots.)^n = c_0 + c_1x + c_2x^2 + \ldots.$$

$$c_0 = {a_0}^n,$$

$$a_0c_1 = na_1c_0,$$

$$2a_0c_2 = (n-1)a_1c_1 + 2na_2c_0,$$

$$3a_0c_3 = (n-2)a_1c_2 + (2n-1)a_2c_1 + 3na_3c_0.$$

. . . . .

. . . . .

*cf.* **6.37.**

**6.362**
$$y = a_1x + a_2x^2 + a_3x^3 + \ldots.$$

$$b_1y + b_2y^2 + b_3y^3 + \ldots. = c_1x + c_2x^2 + c_3x^3 + \ldots$$

$$c_1 = a_1b_1,$$

$$c_2 = a_2b_1 + {a_1}^2b_2,$$

$$c_3 = a_3b_1 + 2a_1a_2b_2 + {a_1}^3b_3,$$

$$c_4 = a_4b_1 + {a_2}^2b_2 + 2a_1a_3b_2 + 3{a_1}^2a_2b_3 + {a_1}^4b_4.$$

. . . .

. . . .

**6.363**
$$e^{a_1x + a_2x^2 + a_3x^3 + \cdots} = 1 + c_1x + c_2x^2 + \ldots$$

$$c_1 = a_1,$$

$$c_2 = a_2 + \frac{1}{2}{a_1}^2,$$

$$c_3 = a_3 + a_1a_2 + \frac{1}{6}a_1^3,$$

$$c_4 = a_4 + a_1a_3 + \frac{1}{2}a_2^2 + \frac{1}{2}a_2a_1^2 + \frac{1}{24}a_1^4.$$

. . . .

. . . .

**6.364**

$$\log(1 + a_1x + a_2x^2 + a_3x^3 + \ldots) = c_1x + c_2x^2 + c_3x^3 + \ldots$$

$$a_1 = c_1,$$

$$2a_2 = a_1c_1 + 2c_2,$$

$$3a_3 = a_2c_1 + 2a_1c_2 + 3c_3,$$

$$4a_4 = a_3c_1 + 2a_2c_2 + 3a_3c_3 + 4a_4.$$

. . .

$$c_1 = a_1,$$

$$c_2 = a_2 - \frac{1}{2}c_1a_1,$$

$$c_3 = a_3 - \frac{1}{3}c_1a_2 - \frac{2}{3}c_2a_1,$$

$$c_4 = a_4 - \frac{1}{4}c_1a_3 - \frac{2}{4}c_2a_2 - \frac{3}{4}c_3a_1.$$

. . .

**6.365**

$$y = a_1x + a_2x^2 + a_3x^3 + \ldots$$

$$z = b_1x + b_2x^2 + b_3x^3 + \ldots$$

$$yz = c_2x^2 + c_3x^3 + c_4x^4 + \ldots$$

$$c_2 = a_1b_1,$$

$$c_3 = a_1b_2 + a_2b_1,$$

$$c_4 = a_1b_3 + a_2b_2 + a_3b_1.$$

. . .

$$c_k = a_1b_{k-1} + a_2b_{k-2} + a_3b_{k-3} + \ldots a_{k-1}b_1.$$

**6.37.** The Multinomial Theorem.

The general term in the expansion of

(1) $$(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots .)^n$$

where $n$ is positive or negative, integral or fractional, is,

(2) $$\frac{n(n-1)(n-2)\ldots(p+1)}{c_1!c_2!c_3!\ldots}\, a_0^p a_1^{c_1} a_2^{c_2} a_3^{c_3} \ldots x^{c_1+2c_2+3c_3+\ldots}$$

where

$$p + c_1 + c_2 + c_3 + \ldots = n.$$

$c_1$, $c_2$, $c_3$, . . . . are positive integers.

If $n$ is a positive integer, and hence $p$ also, the general term in the expansion may be written,

(3) $$\frac{n!}{p!\,c_1!\,c_2!\,\ldots}\,a_0^{p}a_1^{c_1}a_2^{c_2}a_3^{c_3}\ldots\ldots x^{c_1+2c_2+3c_3+}\ldots$$

The coefficient of $x^k$ ($k$ an integer) in the expansion of (1) is found by taking the sum of all the terms (2) or (3) for the different combinations of $p$, $c_1$,$c_2$, $c_3$, . . . . which satisfy

$$c_1 + 2c_2 + 3c_3 + \ldots\ldots = k,$$
$$p + c_1 + c_2 + c_3 + \ldots\ldots = n.$$

cf. **6.361.**

In the following series the coefficients $B_n$ are Bernoulli's numbers (**6.902**) and the coefficients $E_n$, Euler's numbers (**6.903**).

**6.400**

1. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad [x^2 < \infty].$$

2. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad [x^2 < \infty].$$

3. $$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots\ldots$$
$$= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n x^{2n-1} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

4. $$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \frac{1}{4725}x^7 - \ldots\ldots$$
$$= \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}B_n}{(2n)!} x^{2n-1} \qquad [x^2 < \pi^2].$$

5. $$\sec x = 1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4 + \frac{61}{6!}x^6 + \ldots = \sum_{n=0}^{\infty} \frac{E_n}{(2n)!} x^{2n} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

6. $$\csc x = \frac{1}{x} + \frac{1}{3!}x + \frac{7}{3\cdot 5!}x^3 + \frac{31}{3\cdot 7!}x^5 + \ldots$$
$$= \frac{1}{x} + \sum_{n=0}^{\infty} \frac{2(2^{2n+1}-1)}{(2n+2)!} B_{n+1} x^{2n+1} \qquad [x^2 < \pi^2].$$

**6.41**

1. $$\sin^{-1} x = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \ldots\ldots \qquad [x^2 \leqslant 1].$$
$$= \frac{\pi}{2} - \cos^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2\,(2n+1)} x^{2n+1}.$$

2. $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$ (Gregory's Series) $\left[x^2 \leqslant 1\right]$

$$= \frac{\pi}{2} - \cot^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

3. $\tan^{-1} x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\frac{x^2}{1+x^2} + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \ldots\right\}$

$$= \frac{x}{1+x^2}\sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!}\left(\frac{x^2}{1+x^2}\right)^n \qquad x^2 < \infty.$$

4. $\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)x^{2n+1}} \qquad \left[x^2 \geqslant 1\right].$$

5. $\sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2\cdot 3}\frac{1}{x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{1}{x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{1}{x^7} - \ldots$

$$= \frac{\pi}{2} - \csc^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2\,(2n+1)}\, x^{-2n-1} \qquad \left[x > 1\right].$$

## 6.42

1. $(\sin^{-1} x)^2 = x^2 + \frac{2}{3}\frac{x^4}{2} + \frac{2\cdot 4}{3\cdot 5}\frac{x^6}{3} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\frac{x^8}{4} + \ldots$

$$= \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!\,(n+1)}\, x^{2n+2} \qquad \left[x^2 \leqslant 1\right].$$

2. $(\sin^{-1} x)^3 = x^3 + \frac{3!}{5!}3^2\left(1 + \frac{1}{3^2}\right)x^5 + \frac{3!}{7!}3^2 5^2\left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right)x^7 + \ldots$ $\left[x^2 \leqslant 1\right]$.

3. $(\tan^{-1} x)^p = p!\sum_{k_0=1}^{\infty} (-1)^{k_0-1} \frac{x^{2k_0+p-2}}{2k_0+p-2} \prod_{a=1}^{p-1}\left(\sum_{k_a=1}^{ka-1} \frac{1}{2k_a+p-a-2}\right)$.

(Schwatt, Phil. Mag. 31, p. 490, 1916).

4. $\sqrt{1-x^2}\sin^{-1} x = x - \frac{x^3}{3} + \frac{2}{3\cdot 5}x^5 - \frac{2\cdot 4}{3\cdot 5\cdot 7}x^7 + \ldots$

$$= x + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-2}[(n-1)!]^2}{(2n-1)!\,(2n+1)}\, x^{2n+1} \qquad \left[x^2 < 1\right].$$

5. $\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \ldots$

$$= \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!}\, x^{2n+1} \qquad \left[x^2 < 1\right].$$

**6.43**

1. $\log \sin x = \log x - \left\{ \frac{1}{6}x^2 + \frac{1}{180}x^4 + \frac{1}{2835}x^6 + \dots \right\}$

$$= \log x - \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n(2n)!} B_n x^{2n} \qquad \left[x^2 < \pi^2\right].$$

2. $\log \cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{17}{2520}x^8 - \dots$

$$= -\sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)B_n}{n(2n)!} x^{2n} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

3. $\log \tan x = \log x + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \dots$

$$= \log x + \sum_{n=1}^{\infty} \frac{(2^{2n-1}-1)2^{2n}}{n(2n)!} B_n x^{2n} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

4. $\log \cos x = -\frac{1}{2}\left\{ \sin^2 x + \frac{1}{2}\sin^4 x + \frac{1}{3}\sin^6 x + \dots \right\}$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} \sin^{2n} x. \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

**6.44**

1. $\log (1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad \left[-1 < x \leqslant 1\right].$$

$\{\log (1+x)\}^p$ see **7.369.**

2. $\log (x + \sqrt{1+x^2}) = x - \frac{1 \cdot 1}{2 \cdot 3}x^3 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}x^7 + \dots$

$$= x + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{2^{2n-1} n! (n-1)! (2n+1)} x^{2n+1} \qquad \left[-1 \leqslant x \leqslant 1\right].$$

3. $\log (1 + \sqrt{1+x^2}) = \log 2 + \frac{1 \cdot 1}{2 \cdot 2}x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4}x^4 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6}x^6 - \dots$

$$= \log 2 - \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \frac{x^{2n}}{2n} \qquad \left[x^2 \leqslant 1\right].$$

4. $$\log(1+\sqrt{1+x^2}) = \log x + \frac{1}{x} - \frac{1\cdot 1}{2\cdot 3}\frac{1}{x^3} + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{1}{x^5} - \ldots$$
$$= \log x + \frac{1}{x} + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!}{2^{2n-1}n!\,(n-1)!}\,\frac{x^{-2n-1}}{(2n+1)} \qquad \left[x^2 \geqslant 1\right].$$

5. $$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} \qquad \left[0 < x \leqslant 2\right].$$

6. $$\log x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots.$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-1}{x}\right)^n \qquad \left[x \geqslant \tfrac{1}{2}\right].$$

7. $$\log x = 2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right\}$$
$$= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{x-1}{x+1}\right)^{2n+1} \qquad \left[x > 0\right].$$

8. $$\log\frac{1+x}{1-x} = 2\left\{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots.\right\}$$
$$= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1} \qquad \left[x^2 < 1\right].$$

9. $$\log\frac{x+1}{x-1} = 2\left\{\frac{1}{x} + \frac{1}{3}\frac{1}{x^3} + \frac{1}{5}\frac{1}{x^5} + \ldots\right\}$$
$$= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)x^{2n+1}} \qquad \left[x^2 > 1\right].$$

10. $$\sqrt{1+x^2}\log(x+\sqrt{1+x^2}) = x + \frac{1}{3}x^3 - \frac{1\cdot 2}{3\cdot 5}x^5 + \frac{1\cdot 2\cdot 4}{3\cdot 5\cdot 7}x^7 - \ldots$$
$$= x - \sum_{n=1}^{\infty}(-1)^n\frac{(n-1)!\,2^{2n-1}n!}{(2n+1)!}x^{2n+1} \qquad \left[x^2 < 1\right].$$

11. $$\frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} = x - \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \ldots$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}(n!)^2}{(2n+1)!}x^{2n+1} \qquad \left[x^2 < 1\right].$$

12. $$\left\{\log(x+\sqrt{1+x^2})\right\}^2 = \frac{x^2}{1} - \frac{2}{3}\frac{x^4}{2} + \frac{2\cdot 4}{3\cdot 5}\frac{x^6}{3} - \ldots.$$
$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n-2}(n-1)!\,(n-1)!}{(2n-1)!}\frac{x^{2n}}{n}. \qquad \left[x^2 < 1\right]$$

13. $\frac{1}{2}\left\{\log(1+x)\right\}^2 = \frac{1}{2}s_1x^2 - \frac{1}{3}s_2x^3 + \frac{1}{4}s_3x^4 - \ldots$ $\left[x^2<1\right]$.

where $s_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}$ (See **1.876**).

14. $\frac{1}{6}\left\{\log(1+x)\right\}^3 = \frac{1}{3}\cdot\frac{1}{2}s_1x^3 - \frac{1}{4}\left(\frac{1}{2}s_1+\frac{1}{3}s_2\right)x^4$
$+\frac{1}{5}\left(\frac{1}{2}s_1+\frac{1}{3}s_2+\frac{1}{4}s_3\right)x^5 - \ldots$ $\left[x^2<1\right]$.

15. $\frac{\log(1+x)}{(1+x)^n} = x - n(n+1)\left(\frac{1}{n}+\frac{1}{n+1}\right)\frac{x^2}{2!}$
$+n(n+1)(n+2)\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}\right)\frac{x^3}{3!} - \ldots.$ $\left[x^2<1\right]$.

**6.445** (See **6.705**.)

1. $\frac{3}{4x} - \frac{1}{2x^2} + \frac{(1-x)^2}{2x^3}\log\frac{1}{1-x} = \frac{1}{1\cdot2\cdot3} + \frac{x}{2\cdot3\cdot4} + \frac{x^2}{3\cdot4\cdot5} + \ldots.$ $\left[x^2<1\right]$.

2. $\frac{1}{4x}\left\{\frac{1+x}{\sqrt{x}}\log\frac{1+\sqrt{x}}{1-\sqrt{x}} + 2\log(1-x) - 2\right\} = \frac{1}{1\cdot2\cdot3} + \frac{x}{3\cdot4\cdot5}$
$+\frac{x^2}{5\cdot6\cdot7} + \ldots$ $\left[0<x<1\right]$.

3. $\frac{1}{2x}\left\{1 - \log(1+x) - \frac{1-x}{\sqrt{x}}\tan^{-1}x\right\} = \frac{1}{1\cdot2\cdot3} - \frac{x}{3\cdot4\cdot5}$
$+\frac{x^2}{5\cdot6\cdot7} - \ldots$ $\left[0<x\leqslant1\right]$.

**6.455**

1. $-\log(1+x)\cdot\log(1-x) = x^2 + \left(1-\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{2}$
$+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)\frac{x^6}{3} + \ldots.$ $\left[x^2<1\right]$.

2. $\frac{1}{2}\tan^{-1}x\cdot\log\frac{1+x}{1-x} = x^2 + \left(1-\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{3} + \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right)\frac{x^{10}}{5}$
$+\ldots$ $\left[x^2<1\right]$.

3. $\frac{1}{2}\tan^{-1}x\cdot\log(1+x^2) = \left(1+\frac{1}{2}\right)\frac{x^3}{3} - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^5}{5} + \ldots$ $\left[x^2<1\right]$.

**6.456**

1. $\cos\left\{k\log\left(x+\sqrt{1+x^2}\right)\right\} = 1 - \frac{k^2}{2!}x^2 + \frac{k^2(k^2+2^2)}{4!}x^4$
$-\frac{k^2(k^2+2^2)(k^2+4^2)}{6!}x^6 + \ldots.$ $x^2<1$.

$k$ may be any real number.

2. $$\sin\left\{k\log\left(x+\sqrt{1+x^2}\right)\right\} = \frac{k}{1!}x - \frac{k^2(k^2+1^2)}{3!}x^3 + \frac{k^2(k^2+1^2)\,(k^2+3^2)}{5!}x^5 - \ldots \qquad x^2<1.$$

**6.457**

$$\frac{1}{1-2x\cos\alpha+x^2} = 1+\sum_{n=1}^{\infty}A_n x^n. \qquad \left[x^2<1\right],$$

where,

$$A_{2n} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}(2\cos\alpha)^{2k},$$

$$A_{2n+1} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n+k+1}{2k+1}(2\cos\alpha)^{2k+1}.$$

**6.460**

1. $$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \left[x^2<\infty\right].$$

2. $$a^x = 1+x\log a+\frac{(x\log a)^2}{2!}+\frac{(x\log a)^3}{3!}+\ldots \qquad \left[x^2<\infty\right].$$

3. $$e^{e^x} = e\left(1+x+\frac{2}{2!}x^2+\frac{5}{3!}x^3+\frac{15}{4!}x^4+\ldots\right).$$

4. $$e^{\sin x} = 1+x+\frac{x^2}{2!}-\frac{3x^4}{4!}-\frac{8x^5}{5!}+\frac{3x^6}{6!}+\frac{56x^7}{7!}+\ldots$$

5. $$e^{\cos x} = e\left(1-\frac{x^2}{2!}+\frac{4x^4}{4!}-\frac{31x^6}{6!}+\ldots\ldots\right).$$

6. $$e^{\tan x} = 1+x+\frac{x^2}{2!}+\frac{3x^3}{3!}+\frac{9x^4}{4!}+\frac{37x^5}{5!}+\ldots\ldots$$

7. $$e^{\sin^{-1}x} = 1+x+\frac{x^2}{2!}+\frac{2x^3}{3!}+\frac{5x^4}{4!}+\ldots\ldots$$

8. $$e^{\tan^{-1}x} = 1+x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{7x^4}{24}-\ldots\ldots$$

**6.470**

1. $$\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \qquad \left[x^2<\infty\right].$$

2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $\left[x^2 < \infty\right]$.

3. $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n x^{2n-1} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

4. $x \coth x = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \frac{2}{945}x^6 - \cdots$

$$= 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n} B_n}{(2n)!} x^{2n} \qquad \left[x^2 < \pi^2\right].$$

5. $\operatorname{sech} x = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \cdots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{E_n}{(2n)!} x^{2n}$ $\left[x^2 < \frac{\pi}{4}\right]$.

6. $x \operatorname{csch} x = 1 - \frac{1}{6}x^2 + \frac{7}{360}x^4 - \frac{31}{15120}x^6 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2(2^{2n-1}-1)}{(2n)!} B_n x^{2n} \qquad \left[x^2 < \pi^2\right].$$

**6.475**

1. $\cosh x \cos x = 1 - \frac{2^2}{4!}x^4 + \frac{2^4}{8!}x^8 - \frac{2^6}{12!}x^{12} + \cdots$

2. $\sinh x \sin x = \frac{2^2}{2!}x^2 - \frac{2^4}{6!}x^6 + \frac{2^6}{10!}x^{10} - \cdots$

**6.476**

1. $e^{x\cos\theta} \cos(x \sin\theta) = \sum_{n=0}^{\infty} \frac{x^n \cos n\theta}{n!}$ $\left[x^2 < 1\right]$.

2. $e^{x\cos\theta} \sin(x \sin\theta) = \sum_{n=1}^{\infty} \frac{x^n \sin n\theta}{n!}$ $\left[x^2 < 1\right]$.

3. $\cosh(x\cos\theta) \cdot \cos(x\sin\theta) = \sum_{n=0}^{\infty} \frac{x^{2n} \cos 2n\theta}{(2n)!}$ $\left[x^2 < 1\right]$.

4. $\sinh(x\cos\theta) \cdot \cos(x\sin\theta) = \sum_{n=0}^{\infty} \frac{x^{2n+1} \cos(2n+1)\theta}{(2n+1)!}$ $\left[x^2 < 1\right]$.

5. $\cosh(x\cos\theta) \cdot \sin(x\sin\theta) = \sum_{n=0}^{\infty} \frac{x^{2n+1} \sin(2n+1)\theta}{(2n+1)!}$ $\left[x^2 < 1\right]$.

6. $\sinh(x\cos\theta) \cdot \sin(x\sin\theta) = \sum_{n=1}^{\infty} \frac{x^{2n} \sin 2n\theta}{(2n)!}$ $\left[x^2 < 1\right]$.

**6.480**

1. $$\sinh^{-1} x = x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \ldots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2\,(2n+1)}\,x^{2n+1} \qquad \left[x^2<1\right].$$

2. $$\sinh^{-1} x = \log 2x + \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} + \ldots$$
$$= \log 2x + \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 2n}\,x^{-2n} \qquad \left[x^2>1\right].$$

3. $$\cosh^{-1} x = \log 2x - \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} - \ldots$$
$$= \log 2x - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2 2n}\,x^{-2n} \qquad \left[x^2>1\right].$$

4. $$\tanh^{-1} x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \qquad \left[x^2<1\right].$$

5. $$\sinh^{-1}\frac{1}{x} = \frac{1}{x} - \frac{1}{2}\frac{1}{3x^3} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{5x^5} - \ldots.$$
$$= \operatorname{csch}^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2\,(2n+1)}\,x^{-2n-1} \qquad \left[x^2>1\right].$$

6. $$\cosh^{-1}\frac{1}{x} = \log\frac{2}{x} - \frac{1}{2}\frac{x^2}{2} - \frac{1\cdot 3}{2\cdot 4}\frac{x^4}{4} - \ldots$$
$$= \operatorname{sech}^{-1} x = \log\frac{2}{x} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2 2n}\,x^{2n} \qquad \left[x^2<1\right].$$

7. $$\sinh^{-1}\frac{1}{x} = \log\frac{2}{x} + \frac{1}{2}\frac{x^2}{2} - \frac{1\cdot 3}{2\cdot 4}\frac{x^4}{4} + \ldots.$$
$$= \operatorname{csch}^{-1} x = \log\frac{2}{x} + \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 2n}\,x^{2n} \qquad \left[x^2<1\right].$$

8. $$\tanh^{-1}\frac{1}{x} = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \ldots.$$
$$= \coth^{-1} x = \sum_{n=0}^{\infty} \frac{x^{-2n-1}}{2n+1} \qquad \left[x^2>1\right].$$

**6.490**

1. $$\frac{1}{2\sinh x} = \sum_{n=0}^{\infty} e^{-x(2n+1)}.$$

2. $$\frac{1}{2\cosh x} = \sum_{n=0}^{\infty} (-1)^n e^{-x(2n+1)}.$$

3. $$\frac{1}{2}(\tanh x - 1) = \sum_{n=1}^{\infty} (-1)^n e^{-2nx}.$$

4. $$-\frac{1}{2}\log\tanh\frac{x}{2} = \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-x(2n+1)}.$$

**6.491**

$$\frac{1}{2} + \sum_{n=1}^{\infty} e^{-(nx)^2} = \frac{\sqrt{\pi}}{x}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{x}\right)^2}\right\}.$$

By means of this formula a slowly converging series may be transformed into a rapidly converging series.

**6.495**

1. $$\tan x = 2x\left\{\frac{1}{\left(\frac{\pi}{2}\right)^2 - x^2} + \frac{1}{\left(\frac{3\pi}{2}\right)^2 - x^2} + \frac{1}{\left(\frac{5\pi}{2}\right)^2 - x^2} + \cdots\right\}$$
$$= \sum_{n=1}^{\infty} \frac{8x}{(2n-1)^2\pi^2 - 4x^2}.$$

2. $$\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{(2\pi)^2 - x^2} - \frac{2x}{(3\pi)^2 - x^2} - \cdots = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 - x^2}.$$

3. $$\sec x = \frac{\pi}{\left(\frac{\pi}{2}\right)^2 - x^2} - \frac{3\pi}{\left(\frac{3\pi}{2}\right)^2 - x^2} + \frac{5\pi}{\left(\frac{5\pi}{2}\right)^2 - x^2} - \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4(2n-1)\pi}{(2n-1)^2\pi^2 - 4x^2}.$$

4. $$\csc x = \frac{1}{x} + \frac{2x}{\pi^2 - x^2} - \frac{2x}{(2\pi)^2 - x^2} + \frac{2x}{(3\pi)^2 - x^2} - \cdots$$
$$= \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2x}{n^2\pi^2 - x^2}.$$

By replacing $x$ by $ix$ the corresponding series for the hyperbolic functions may be written.

## INFINITE PRODUCTS

### 6.50

1. $\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$.

2. $\sinh x = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2\pi^2}\right)$.

3. $\cos x = \prod_{n=0}^{\infty} \left(1 - \frac{4x^2}{(2n+1)^2\pi^2}\right)$.

4. $\cosh x = \prod_{n=0}^{\infty} \left(1 + \frac{4x^2}{(2n+1)^2\pi^2}\right)$.

### 6.51

1. $\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos \frac{x}{2^n}$.

### 6.52

1. $\frac{1}{1-x} = \prod_{n=0}^{\infty} (1 + x^{2n})$. $[x^2 < 1]$.

### 6.53

1. $\cosh x - \cos y = 2\left(1 + \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{(2n\pi + y)^2}\right)\left(1 + \frac{x^2}{(2n\pi - y)^2}\right)$.

2. $\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(2n\pi + y)^2}\right)\left(1 - \frac{x^2}{(2n\pi - y)^2}\right)$.

**6.55** The convergent infinite series:

$$1 + u_1 + u_2 + \ldots = 1 + \sum_{n=1}^{\infty} u_n.$$

may be transformed into the infinite product

$$(1 + v_1)(1 + v_2)(1 + v_3) \ldots \ldots$$
$$= \prod_{n=1}^{\infty} (1 + v_n),$$

where

$$v_n = \frac{u_n}{1 + u_1 + u_2 + \ldots + u_{n-1}}.$$

**6.600** The Gamma Function:

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^z}{1 + \frac{z}{n}},$$

$z$ may have any real or complex value, except $0, -1, -2, -3, \ldots \ldots$

**6.601**

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

**6.602**

$$\gamma = \underset{m \to \infty}{\text{Limit}} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \ldots \ldots + \frac{1}{m} - \log m \right\}$$
$$= \int_0^{\infty} \left\{ \frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-t}}{t} \right\} dt = 0.5772157 \ldots$$

**6.603**

$$\Gamma(z + 1) = z\Gamma(z),$$
$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}.$$

**6.604** For $z$ real and positive $= x$:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt,$$

$$\log \Gamma(1 + x) = \left(x + \frac{1}{2}\right) \log x - x + \frac{1}{2} \log 2\pi + \int_0^{\infty} \left\{ \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right\} e^{-xt} \frac{dt}{t}.$$

**6.605** If $z = n$, a positive integer:

$$\Gamma(n) = (n - 1)!,$$
$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \ldots \ldots (2n - 1)}{2^n} \sqrt{\pi},$$
$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

**6.606** The Beta Function. If $x$ and $y$ are real and positive:

$$\mathrm{B}(x, y) = \mathrm{B}(y, x) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},$$

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

$$\mathrm{B}(x+1, y) = \frac{x}{x+y}\,\mathrm{B}(x, y),$$

$$\mathrm{B}(x, 1-x) = \frac{\pi}{\sin \pi x}.$$

**6.610** For $x$ real and positive:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right).$$

**6.611**

$$\psi(x+1) = \frac{1}{x} + \psi(x),$$

$$\psi(1-x) = \psi(x) + \pi \cot \pi x.$$

**6.612**

$$\psi(\tfrac{1}{2}) = -\gamma - 2\log 2,$$
$$\psi(1) = -\gamma,$$
$$\psi(2) = 1 - \gamma,$$
$$\psi(3) = 1 + \frac{1}{2} - \gamma,$$
$$\psi(4) = 1 + \frac{1}{2} + \frac{1}{3} - \gamma.$$

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**6.613**

$$\psi(x) = \int_0^{\infty}\left\{\frac{e^{-t}}{t} - \frac{e^{-tx}}{1-e^{-t}}\right\}dt$$

$$= -\gamma + \int_0^1 \frac{1 - t^{x-1}}{1-t}\,dt.$$

**6.620**

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}$$
$$= \frac{1}{2}\left\{ \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) \right\}.$$

**6.621**

$$\beta(x+1) + \beta(x) = \frac{1}{x},$$
$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}.$$

**6.622**

$$\beta(1) = \log 2,$$
$$\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}.$$

**6.630** Gauss's $\Pi$ Function:

1. $\Pi(k, z) = k^z \prod_{n=1}^{k} \frac{n}{z+n}.$
2. $\Pi(k, z+1) = \Pi(k, z) \cdot \frac{1+z}{1+\frac{1+z}{k}}.$
3. $\Pi(z) = \underset{k \to \infty}{\text{Limit}}\, \Pi(k, z).$
4. $\Pi(z) = \Gamma(z+1).$
5. $\Pi(-z)\,\Pi(z-1) = \pi \csc \pi z.$
6. $\Pi\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$

**6.631** If $z$ is an integer, $n$,

$$\Pi(n) = n!$$

DEFINITE INTEGRALS EXPRESSED AS INFINITE SERIES

**6.700**

$$\int_0^x e^{-x^2}\,dx = \sum_{k=0}^{\infty} \frac{(-1)k}{k!(2k+1)} x^{2k+1}.$$
$$= e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k+1)}.$$

Darling (Quarterly Journal, 49, p. 36, 1920) has obtained an approximation to this integral:

$$\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \tan^{-1} \left\{ e^{\sqrt{\pi}}(1 + x^2 e^{-\sqrt{\pi}})^2 \right\}^{-x}$$

Fresnel's Integrals:

**6.701**
$$\int_0^x \cos(x^2)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\,(4k+1)} x^{4k+1}$$

$$= \cos(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{4k+1}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (4k+1)}$$

$$+ \sin(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1} x^{4k+3}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (4k+3)}.$$

**6.702**
$$\int_0^x \sin(x^2)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!\,(4k+3)} x^{4k+3}$$

$$= \sin(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k}}{1 \cdot 3 \cdot 5 \ldots (4k+1)} x^{4k+1}$$

$$- \cos(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1} x^{4k+3}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (4k+3)}.$$

**6.703**
$$\int_0^1 \frac{t^{a-1}}{1+t^b} dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{a+nb}$$

**6.704**
$$\frac{1}{(k-1)!} \int_0^1 \frac{t^{a-1}(1-t)^{k-1}}{1-xt^b} dt$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(a+nb)\,(a+nb+1)\,(a+nb+2) \ldots (a+nb+k-1)} \qquad [b>0,\ x^2 \leqslant 1].$$

(Special cases, **6.445** and **6.922**).

**6.705**
$$\int_0^x e^{-t} t^{y-1} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+y}}{n!(n+y)} = e^{-x} \sum_{n=0}^{\infty} \frac{x^{n+y}}{y(y+1) \ldots (y+n)}.$$

**6.706** If the sum of the series,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad [0 < x < 1]$$

is known, then

$$\sum_{n=0}^{\infty} \frac{c_n x^n}{(a+nb)\,(a+nb+1)\,(a+nb+2) \ldots (a+nb+k-1)} \qquad [b>0]$$

$$= \frac{1}{(k-1)!} \int_0^1 t^{a-1}(1-t)^{k-1} f(xt^b) dt.$$

**6.707** $$\int_0^\infty f(x)\sum_{n=1}^{\infty}\frac{1}{n}\sin nx\cdot dx=\frac{1}{2}\int_0^{2\pi}(\pi-t)\sum_{n=0}^{\infty}f(t+2n\pi)\cdot dt.$$

Example 1. $f(x)=e^{-kx}$ $[k>0]$.

1. $$\frac{1}{k}+2k\sum_{n=1}^{\infty}\frac{1}{k^2+n^2}=\pi\frac{e^{k\pi}+e^{-k\pi}}{e^{k\pi}-e^{-k\pi}}.$$

Replacing $k$ by $\frac{k}{2}$, and subtracting,

2 $$\frac{1}{k}+2k\sum_{n=1}^{\infty}(-1)^n\frac{1}{k^2+n^2}=\frac{2\pi}{e^{k\pi}-e^{-k\pi}}.$$

Example 2. With $f(x)=e^{-\lambda x}\cos\mu x$ and $e^{-\lambda x}\sin\mu x$.

3. $$\frac{\lambda}{\lambda^2+\mu^2}+\sum_{n=1}^{\infty}\left\{\frac{\lambda}{\lambda^2+(n-\mu)^2}+\frac{\lambda}{\lambda^2+(n+\mu)^2}\right\}=\frac{\pi\sinh 2\lambda\pi}{\cosh 2\lambda\pi-\cos 2\mu\pi}.$$

4. $$\frac{\mu}{\lambda^2+\mu^2}-\sum_{n=1}^{\infty}\left\{\frac{n-\mu}{\lambda^2+(n-\mu)^2}+\frac{n+\mu}{\lambda^2+(n+\mu)^2}\right\}=\frac{\pi\sin 2\mu\pi}{\cosh 2\lambda\pi-\cos 2\mu\pi}.$$

**6.709** If the sum of the series,

$$f(x)=\sum_{n=0}^{\infty}a_nx^n,$$

is known, then

$$a_0+a_1y+a_2y(y+1)+a_3y(y+1)(y+2)+\cdots\cdots=\frac{\int_0^\infty e^{-t}t^{y-1}f(t)\,dt}{\Gamma(y)}.$$

**6.710** The complete elliptic integral of the first kind:

$$K=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\cdots\cdot\right\}$$

$$=\frac{\pi}{2}\left\{1+\sum_{n=1}^{\infty}\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots\cdot2n}\right)^2k^{2n}\right\}\qquad[k^2<1].$$

If $$k'=\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}$$

$$K=\frac{\pi(1+k')}{2}\left\{1+\left(\frac{1}{2}\right)^2k'^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2k'^4+\cdots\right\}$$

$$=\frac{\pi(1+k')}{2}\left\{1+\sum_{n=1}^{\infty}\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots\cdot2n}\right)^2k'^{2n}\right\}.$$

**6.711** The complete elliptic integral of the second kind:

$$E = \int^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta.$$

$$E = \frac{\pi}{2}\left\{ 1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \ldots . \right\}.$$

$$= \frac{\pi}{2}\left\{ 1 - \sum_{n=1}^{\infty} \left(\frac{1\cdot 3\cdot 5 \ldots (2n-1)}{2\cdot 4\cdot 6 \ldots . 2n}\right)^2 \frac{k^{2n}}{2n-1} \right\}.$$

If $$k' = \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}.$$

$$E = \frac{\pi(1-k')}{2}\left\{ 1 + 5\left(\frac{1}{2}\right)^2 k'^2 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k'^4 + \ldots \right\}$$

$$= \frac{\pi(1-k')}{2}\left\{ 1 + \sum_{n=1}^{\infty} (4n+1)\left(\frac{1\cdot 3\cdot 5 \ldots (2n-1)}{2\cdot 4\cdot 6 \ldots 2n}\right)^2 k'^{2n} \right\}$$

$$= \frac{\pi}{2(1+k')}\left\{ 1 + \left(\frac{1}{2}\right)^2 k'^2 + \left(\frac{1}{2\cdot 4}\right)^2 k'^4 + \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}\right)^2 k'^6 + \ldots \right\}$$

$$= \frac{\pi}{2(1+k')}\left\{ 1 + k'^2\left[\frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{1\cdot 3 \ldots . (2n-1)}{2\cdot 4\cdot 6 \ldots (2n+2)}\right)^2 k'^{2n}\right] \right\}.$$

## FOURIER'S SERIES

**6.800** If $f(x)$ is uniformly convergent in the interval:

$$-c < x < +c$$

$$f(x) = \frac{1}{2}b_0 + b_1 \cos\frac{\pi x}{c} + b_2 \cos\frac{2\pi x}{c} + b_3 \cos\frac{3\pi x}{c} + \ldots .$$

$$+ a_1 \sin\frac{\pi x}{c} + a_2 \sin\frac{2\pi x}{c} + a_3 \sin\frac{3\pi x}{c} + \ldots .$$

$$b_m = \frac{1}{c}\int_{-c}^{+c} f(x) \cos\frac{m\pi x}{c}\, dx,$$

$$a_m = \frac{1}{c}\int_{-c}^{+c} f(x) \sin\frac{m\pi x}{c}\, dx.$$

**6.801** If $f(x)$ is uniformly convergent in the interval:

$$0 < x < c$$

$$f(x) = \frac{1}{2}b_0 + b_1 \cos\frac{2\pi x}{c} + b_2 \cos\frac{4x\pi}{c} + b_3 \cos\frac{6\pi x}{c} + \ldots .$$

$$+ a_1 \sin\frac{2\pi x}{c} + a_2 \sin\frac{4\pi x}{c} + a_3 \sin\frac{6\pi x}{c} + \ldots .$$

$$b_m = \frac{2}{c}\int_{0}^{c} f(x) \cos\frac{2m\pi x}{c}\, dx,$$

$$a_m = \frac{2}{c}\int_{0}^{c} f(x) \sin\frac{2m\pi x}{c}\, dx.$$

**6.802** Special Developments in Fourier's Series.

$$f(x) = a \text{ from } x = kc \text{ to } x = (k + \tfrac{1}{2})c,$$

$$f(x) = -a \text{ from } x = (k + \tfrac{1}{2})c \text{ to } x = (k + 1)c,$$

where $k$ is any integer, including 0.

$$f(x) = \frac{4a}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi}{c} x.$$

**6.803**

$$\begin{aligned} f(x) &= mx, && -\frac{c}{4} \leqslant x \leqslant +\frac{c}{4} \\ &= -m\left(x - \frac{c}{2}\right), && \frac{c}{4} \leqslant x \leqslant \frac{3c}{4} \\ &= m(x - c), && \frac{3c}{4} \leqslant x \leqslant \frac{5c}{4} \\ &= -m\left(x - \frac{3c}{2}\right), && \frac{5c}{4} \leqslant x \leqslant \frac{7c}{4} \end{aligned}$$

. . . . .

. . . . .

$$f(x) = \frac{2mc}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)^2} \sin \frac{2(2n-1)\pi}{c} x.$$

**6.804**

$$\begin{aligned} f(x) &= mx, && -\frac{c}{2} < x < +\frac{c}{2} \\ &= m(x - c), && +\frac{c}{2} < x < \frac{3c}{2}, \end{aligned}$$

$$f(x) = \frac{cm}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{2n\pi x}{c}.$$

**6.805**

$$\begin{aligned} f(x) &= -a, && -5b \leqslant x \leqslant -3b, \\ &= \frac{a}{b}(x + 2b), && -3b \leqslant x \leqslant -b, \\ &= a, && -b \leqslant x \leqslant +b, \\ &= -\frac{a}{b}(x - 2b), && b \leqslant x \leqslant 3b, \\ &= -a, && 3b \leqslant x \leqslant 5b. \end{aligned}$$

. . . . .

. . . . .

$$f(x) = \frac{8\sqrt{2}a}{\pi^2} \left\{ \cos \frac{\pi x}{4b} - \frac{1}{3^2} \cos \frac{3\pi x}{4b} - \frac{1}{5^2} \cos \frac{7\pi x}{4b} + \frac{1}{7^2} \cos \frac{7\pi x}{4b} + \cdots \right\}$$

6.806
$$f(x) = \frac{b}{l}x + b, \qquad -l \leqslant x \leqslant 0,$$
$$= -\frac{b}{l}x + b, \qquad 0 \leqslant x \leqslant l.$$
$$f(x) = \frac{8b}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\frac{\pi x}{2l}.$$

6.807
$$f(x) = \frac{a}{b}x, \qquad 0 \leqslant x \leqslant b,$$
$$= -\frac{a}{l-b}x + \frac{al}{l-b}, \qquad b \leqslant x \leqslant l,$$
$$f(x) = \frac{2al^2}{\pi^2 b(l-b)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi b}{l}\sin\frac{n\pi x}{l}.$$

6.810 $$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx \qquad [-\pi < x < \pi].$$

6.811 $$\cos ax = \frac{2}{\pi}\sin a\pi\left\{\frac{1}{2a} + a\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2 - a^2}\cos nx\right\} \qquad [-\pi < x < \pi].$$

6.812 $$\sin ax = \frac{2}{\pi}\sin a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2 - a^2}n\sin nx \qquad [-\pi < x < \pi].$$

6.813 $$\frac{\pi - x}{2} = \sum_{n=1}^{\infty}\frac{\sin nx}{n} \qquad [0 < x < 2\pi].$$

6.814 $$\frac{1}{2}\log\frac{1}{2(1-\cos x)} = \sum_{n=1}^{\infty}\frac{\cos nx}{n} \qquad [0 < x < 2\pi].$$

6.815 $$\frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \qquad [0 < x < 2\pi].$$

6.816 $$\frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} = \sum_{n=1}^{\infty}\frac{\sin nx}{n^3} \qquad [0 < x < 2\pi].$$

6.817 $$\frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48} = \sum_{n=1}^{\infty}\frac{\cos nx}{n^4} \qquad [0 < x < 2\pi].$$

6.818 $$\frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240} = \sum_{n=1}^{\infty}\frac{\sin nx}{n^5} \qquad [0 < x < 2\pi].$$

6.820 $$x^2 = \frac{c^2}{3} - \frac{4c^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos\frac{n\pi x}{c} \qquad \left[-c \leqslant x \leqslant c\right].$$

6.821 $$\frac{e^x}{e^c - e^{-c}} = \frac{1}{2c} - c\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n\pi)^2 + c^2} \cos\frac{n\pi x}{c} + \pi\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n\pi)^2 + c^2} \sin\frac{n\pi x}{c} \qquad \left[-c < x < c\right].$$

6.822 $$e^{cx} = \frac{2c}{\pi}(e^{c\pi} - 1)\left\{\frac{1}{2c^2} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{c^2 + n^2} \cos nx\right\} \qquad \left[0 < x < \pi\right].$$

6.823 $$\cos 2x - \left(\frac{\pi}{2} - x\right)\sin 2x + \sin^2 x \log(4\sin^2 x) = \sum_{n=1}^{\infty} \frac{\cos 2(n+1)x}{n(n+1)} \qquad \left[0 \leqslant x \leqslant \pi\right].$$

6.824 $$\sin 2x - (\pi - 2x)\sin^2 x - \sin x \cos x \log(4\sin^2 x) = \sum_{n=1}^{\infty} \frac{\sin 2(n+1)x}{n(n+1)} \qquad \left[0 \leqslant x \leqslant \pi\right].$$

6.825 $$\frac{1}{2} - \frac{\pi}{4}\sin x = \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n-1)(2n+1)} \qquad \left[0 \leqslant x \leqslant \frac{\pi}{2}\right].$$

6.830 $$\frac{r\sin x}{1 - 2r\cos x + r^2} = \sum_{n=1}^{\infty} r^n \sin nx \qquad \left[r^2 < 1\right].$$

6.831 $$\tan^{-1}\frac{r\sin x}{1 - r\cos x} = \sum_{n=1}^{\infty} \frac{1}{n} r^n \sin nx \qquad \left[r < 1\right].$$

6.832 $$\frac{1}{2}\tan^{-1}\frac{2r\sin x}{1 - r^2} = \sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} \sin(2n-1)x \qquad \left[r^2 < 1\right].$$

6.833 $$\frac{1 - r\cos x}{1 - 2r\cos x + r^2} = \sum_{n=0}^{\infty} r^n \cos nx \qquad \left[r^2 < 1\right].$$

6.834 $$\log\frac{1}{\sqrt{1 - 2r\cos x + r^2}} = \sum_{n=1}^{\infty} \frac{1}{n} r^n \cos nx \qquad \left[r^2 < 1\right].$$

**6.835** $$\frac{1}{2}\tan^{-1}\frac{2r\cos x}{1-r^2}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{r^{2n-1}}{2n-1}\cos(2n-1)x \qquad \left[r^2<1\right].$$

## NUMERICAL SERIES

**6.900**

$$S_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots=\sum_{k=1}^{\infty}\frac{1}{k^n},$$

$S_1=\infty$

$S_2=\frac{\pi^2}{6}=1.6449340668$

$S_3=\frac{\pi^3}{25.79436\ldots}=1.2020569032$

$S_4=\frac{\pi^4}{90}=1.0823232337$

$S_5=\frac{\pi^5}{295.1215}=1.0369277551$

$S_6=\frac{\pi^6}{945}=1.0173430620,$

$S_7=\frac{\pi^7}{2995.286\ldots}=1.0083492774$

$S_8=\frac{\pi^8}{9450}=1.0040773562,$

$S_9=\frac{\pi^9}{29749.35\ldots}=1.0020083928,$

$S_{10}=1.000994575I,$
$S_{11}=1.0004941886.$

**6.901**

$$u_n=1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\ldots=\sum_{k=0}^{\infty}(-1)^{k-1}\frac{1}{(2k+1)^n},$$

$u_1=\frac{\pi}{4},$

$u_2=0.9159656\ldots$

$u_4=0.98894455\ldots$

$u_6=0.99868522\ldots$

A table of $u_n$ from $n=1$ to $n=38$ to 18 decimal places is given by Glaisher, Messenger of Mathematics, 42, p. 49, 1913.

**6.902** Bernoulli's Numbers.

1. $$\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n=\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\ldots=\sum_{k=1}^{\infty}\frac{1}{k^{2n}}.$$

2. $$\frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}B_n=\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\ldots=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n}}.$$

3. $$\frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}B_n=\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\ldots=\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^{2n}}.$$

$B_1=\frac{1}{6},$

$B_2=\frac{1}{30},$

$B_3=\frac{1}{42},$

$B_4=\frac{1}{30},$

$$B_5 = \frac{5}{66}, \qquad B_8 = \frac{3617}{510},$$

$$B_6 = \frac{691}{2730} \qquad B_9 = \frac{43867}{798},$$

$$B_7 = \frac{7}{6}, \qquad B_{10} = \frac{174611}{330}.$$

**6.903** Euler's Numbers

$$\frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_n = 1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \quad \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(2k-1)^{2n+1}}.$$

$$E_1 = 1, \qquad E_4 = 1385,$$

$$E_2 = 5, \qquad E_5 = 50521,$$

$$E_3 = 61, \qquad E_6 = 2702765.$$

**6.904**

$$E_n - \frac{2n(2n-1)}{2!} E_{n-1} + \frac{2n(2n-1)(2n-2)(2n \quad 3)}{4!} E_{n-2} - \ldots .$$

$$- \ldots \ldots + (-1)^n = 0.$$

**6.905**

$$\frac{2^{2n}(2^{2n}-1)}{2n} B_n = (2n-1)E_{n-1} - \frac{(2n-1)(2n-2)(2n-3)}{3!} E_{n-2}$$

$$+ \frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)}{5!} E_{n-3} - \ldots \ldots + (-1)^{n-1}.$$

**6.910**

$$S_r = \sum_{n=1}^{\infty} \frac{n^r}{n!}$$

$$S_1 = e, \qquad S_5 = 52e,$$

$$S_2 = 2e, \qquad S_6 = 203e,$$

$$S_3 = 5e, \qquad S_7 = 877e,$$

$$S_4 = 15e, \qquad S_8 = 4140e.$$

**6.911**

$$S_r = \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^r}.$$

$$S_1 = \frac{1}{2}, \qquad S_3 = \frac{32 - 3\pi^2}{64},$$

$$S_2 = \frac{\pi^2 - 8}{16}, \qquad S_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}.$$

**6.912**

1. $\log 2 = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$.

2. $\frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$.

**6.913**

1. $2\log 2 - 1 = \sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)}$.

2. $\frac{3}{2}(\log 3 - 1) = \sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)}$.

3. $-3 + \frac{3}{2}\log 3 + 2\log 2 = \sum_{n=1}^{\infty} \frac{1}{n(36n^2-1)}$.

**6.914**

$$S_r = \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}\right)^2 \frac{1}{2n+r},$$

$u_2 = 0.9159656\ldots$ (see **6.901**)

$S_0 = 2\log 2 - \frac{4}{\pi}u_2$,

$S_1 = \frac{4}{\pi}u_2 - 1$,

$S_2 = \frac{2}{\pi} - \frac{1}{2}$,

$S_3 = \frac{1}{2\pi}(2u_2 + 1) - \frac{1}{3}$,

$S_4 = \frac{10}{9\pi} - \frac{1}{4}$,

$S_5 = \frac{1}{32\pi}(18u_2 + 13) - \frac{1}{5}$,

$S_6 = \frac{178}{225\pi} - \frac{1}{6}$,

$S_7 = \frac{1}{128\pi}(50u_2 + 43) - \frac{1}{7}$,

$S_{-1} = 1 - \frac{2}{\pi}$,

$S_{-2} = \frac{1}{2}\log 2 + \frac{1}{4} - \frac{1}{2\pi}(2u_2 + 1)$,

$S_{-3} = \frac{1}{3} - \frac{10}{9\pi}$,

$S_{-4} = \frac{9}{32}\log 2 + \frac{11}{128} - \frac{1}{32\pi}(18u_2 + 13)$,

$S_{-5} = \frac{1}{5} - \frac{178}{225\pi}$,

$S_{-6} = \frac{25}{128}\log 2 + \frac{71}{1536} - \frac{1}{128\pi}(50u_2 + 43)$.

When $r$ is a negative even integer the value $n = \frac{r}{2}$ is to be excluded in the summation.

**6.915**

1. $A_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} = \frac{(2n-1)!}{2^{2n-1} n!(n-1)!}$.

2. $1 - \frac{\pi}{4} = \sum_{n=1}^{\infty} A_n \frac{1}{4n^2-1}$.

3. $\frac{\pi}{2} - 1 = \sum_{n=1}^{\infty} A_n \frac{1}{2n+1}$.

4. $\log(1+\sqrt{2}) - 1 = \sum_{n=1}^{\infty} (-1)^n A_n \frac{1}{2n+1}$.

5. $\frac{1}{2} = \sum_{n=1}^{\infty} {A_n}^2 \frac{4n+1}{(2n-1)(2n+2)}$.

6. $\frac{2}{\pi} - \frac{1}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} {A_n}^3 \frac{4n+1}{(2n-1)(2n+2)}$.

7. $\frac{2}{\pi} - 1 = \sum_{n=1}^{\infty} (-1)^n {A_n}^3 (4n+1)$.

8. $\frac{1}{2} - \frac{4}{\pi^2} = \sum_{n=1}^{\infty} {A_n}^4 \frac{4n+1}{(2n-1)(2n+2)}$.

**6.916**

If $m$ is an integer, and $n = m$ is excluded from the summation:

1. $-\frac{3}{4m^2} = \sum_{n=1}^{\infty} \frac{1}{m^2 - n^2}$.

2. $\frac{3}{4m^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{m^2 - n^2}$. ($m$ even)

**6.917**

1. $1 = \sum_{n=2}^{\infty} \frac{n-1}{n!}$.

2. $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

3. $2 \log 2 = \sum_{n=1}^{\infty} \frac{12n^2 - 1}{n(4n^2-1)^2}$.

**6.918** $\frac{2}{\sqrt{3}} \log \frac{1+\sqrt{3}}{\sqrt{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)} \frac{1}{2^n}$.

**6.919** $\frac{1}{2}(1 - \log 2) = \sum_{n=1}^{\infty} \left\{ n \log\left(\frac{2n+1}{2n-1}\right) - 1 \right\}$.

**6.920**

1. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2.71828$.

2. $\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \ldots . \quad = 0.36788.$

3. $\frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots . = 1.54308.$

4. $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \ldots . = 1.175201.$

5. $\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \ldots . \quad = 0.54030.$

6. $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \ldots . \quad = 0.84147.$

**6.921**

1. $\frac{4}{5} = 1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \ldots .$

2. $\frac{9}{10} = 1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \ldots .$

3. $\frac{16}{17} = 1 - \frac{1}{4^2} + \frac{1}{4^4} - \frac{1}{4^6} + \ldots .$

4. $\frac{25}{26} = 1 - \frac{1}{5^2} + \frac{1}{5^4} - \frac{1}{5^6} + \ldots .$

**6.922** $\frac{(2^{\frac{1}{4}} - 1)\Gamma(\frac{1}{4})}{2^{\frac{1}{4}}\pi^{\frac{3}{4}}} = e^{-\pi} + e^{-9\pi} + e^{-25\pi} + \ldots ; \Gamma(\tfrac{1}{4}) = 3.6256 \ldots .$

**6.923** (Special cases of **6.705**):

1. $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \ldots \quad = \log 2 - \frac{1}{2}.$

2. $\frac{1}{1\cdot 2\cdot 3} - \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} - \ldots \quad = \frac{1}{2}(1 - \log 2).$

3. $\frac{1}{2\cdot 3\cdot 4} + \frac{1}{4\cdot 5\cdot 6} + \frac{1}{6\cdot 7\cdot 8} + \ldots \quad = \frac{3}{4} - \log 2.$

4. $\frac{1}{2\cdot 3\cdot 4} - \frac{1}{4\cdot 5\cdot 6} + \frac{1}{6\cdot 7\cdot 8} - \ldots \quad = \frac{1}{4}(\pi - 3).$

5. $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{4\cdot 5\cdot 6} + \frac{1}{7\cdot 8\cdot 9} + \ldots \quad = \frac{1}{4}\left(\frac{\pi}{\sqrt{3}} - \log 3\right).$

6. $\frac{1}{2\cdot 3\cdot 4} + \frac{1}{6\cdot 7\cdot 8} + \frac{1}{10\cdot 11\cdot 12} + \ldots . \quad = \frac{\pi}{8} - \frac{1}{2}\log 2.$

7. $\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{4\cdot 5\cdot 6\cdot 7} + \frac{1}{7\cdot 8\cdot 9\cdot 10} + \ldots = \frac{1}{6}\left(1 + \frac{\pi}{2\sqrt{3}}\right) - \frac{1}{4}\log 3.$

# VII. SPECIAL APPLICATIONS OF ANALYSIS.

## 7.10 Indeterminate Forms.

**7.101** $\frac{0}{0}$. If $\frac{f(x)}{F(x)}$ assumes the indeterminate value $\frac{0}{0}$ for $x = a$, the true value of the quotient may be found by replacing $f(x)$ and $F(x)$ by their developments in series, if valid for $x = a$.

Example:

$$\left[\frac{\sin^2 x}{1 - \cos x}\right]_{x=0};$$

$$\frac{\sin^2 x}{1 - \cos x} = \frac{\left(x - \frac{x^3}{3!} + \cdots\right)^2}{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots} = \frac{\left(1 - \frac{x^2}{3!} + \cdots\right)^2}{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}$$

Therefore,

$$\left[\frac{\sin^2 x}{1 - \cos x}\right]_{x=0} = 2.$$

**7.102** L'Hospital's Rule. If $f(a + h)$ and $F(a + h)$ can be developed by Taylor's Theorem (**6.100**) then the true value of $\frac{f(x)}{F(x)}$ for $x = a$ is,

$$\frac{f'(a)}{F'(a)}$$

provided that this has a definite value (0, finite, or infinite). If the ratio of the first derivatives is still indeterminate, the true value may be found by taking that of the ratio of the first one of the higher derivatives that is definite.

**7.103** The true value of $\frac{f(x)}{F(x)}$ for $x = a$ is the limit, for $h = 0$, of

$$\frac{q!}{p!} h^{p-q} \frac{f^{(p)}(a)}{F^{(q)}(a)}$$

where $f^{(p)}(a)$ and $F^{(q)}(a)$ are the first of the higher derivatives of $f(x)$ and $F(x)$ that do not vanish for $x = a$. The true value of $\frac{f(x)}{F(x)}$ for $x = a$ is 0 if $p > q$, $\infty$ if $p < q$, and equal to $\frac{f^{(p)}(a)}{F^{(p)}(a)}$ if $p = q$.

Example:

$$\left[\frac{\sinh x - x\cosh x}{\sin x - x\cos x}\right]_{x=0} = \left[\frac{-x\sinh x}{x\sin x}\right]_{x=0}.$$

$$= \left[-\frac{\sinh x}{\sin x}\right]_{x=0} = \left[-\frac{\cosh x}{\cos x}\right]_{x=0} = -1.$$

**7.104** Failure of L'Hospital's Rule. In certain cases this rule fails to determine the true value of an expression for the reason that all the higher derivatives vanish at the limit. In such cases the true value may often be found by factoring the given expression, or resolving into partial fractions (**1.61**).

Example:

$$\left[\frac{\sqrt{x^2 - a^2}}{\sqrt{x - a}}\right]_{x=a} = \left[\sqrt{x + a}\right]_{x=a} = \sqrt{2a}.$$

**7.105** In applying L'Hospital's Rule, if any of the successive quotients contains a factor which can be evaluated at once its determinate value may be substituted.

Example:

$$\left[\frac{(1 - x)e^x - 1}{\tan^2 x}\right]_{x=0} = \left[\frac{-xe^x}{2\tan x\sec^2 x}\right]_{x=0}$$

$$\left[\frac{x}{\tan x}\right]_{x=0} = 1.$$

Hence the given function is,

$$\left[-\frac{e^x}{2\sec^2 x}\right]_{x=0} = -\frac{1}{2}.$$

**7.106** If the given function can be separated into factors each of which is indeterminate, the factors may be evaluated separately.

Example:

$$\left[\frac{(e^x - 1)\tan^2 x}{x^3}\right]_{x=0} = \left[\left(\frac{\tan x}{x}\right)^2 \frac{e^x - 1}{x}\right]_{x=0} = 1.$$

**7.110** $\frac{\infty}{\infty}$. If, for $x = a$, $\frac{f(x)}{F(x)}$ takes the form $\frac{\infty}{\infty}$, this quotient may be written:

$$\frac{\frac{1}{F(x)}}{\frac{1}{f(x)}}$$

which takes the form $\frac{0}{0}$ for $x = a$ and the preceding sections will apply to it.

**7.111** L'Hospital's Rule (**7.102**) may be applied directly to indeterminate forms $\frac{\infty}{\infty}$, if the expansion by Taylor's Theorem is valid.

Example:

$$\left[\frac{x}{e^x}\right]_{x=\infty} = \left[\frac{1}{e^x}\right]_{x=\infty} = 0.$$

**7.112** If $f(x)$ and $x$ approach $\infty$ together, and if $f(x+1) - f(x)$ approaches a definite limit, then,

$$\operatorname*{Limit}_{x\to\infty}\left[\frac{f(x)}{x}\right] = \operatorname*{Limit}_{x\to\infty}\Big[f(x+1) - f(x)\Big].$$

**7.120** $0 \times \infty$ . . If, for $x = a$, $f(x) \times F(x)$ takes the form $0 \times \infty$, this product may be written,

$$\frac{f(x)}{\dfrac{1}{F(x)}}$$

which takes the form $\frac{0}{0}$ (**7.101**).

**7.130** $\infty - \infty$. If, $\operatorname*{Limit}_{x\to a} f(x) = \infty$ and $\operatorname*{Limit}_{x\to\infty} F(x) = \infty$,

$$f(x) - F(x) = f(x)\left\{1 - \frac{F(x)}{f(x)}\right\}.$$

If $\operatorname*{Limit}_{x\to\infty} \dfrac{F(x)}{f(x)}$ is different from unity the true value of $f(x) - F(x)$ for $x = a$ is $\infty$.

If $\operatorname*{Limit}_{x\to\infty} \dfrac{F(x)}{f(x)} = +1$, the expression has the indeterminate form $\infty \times 0$ which may be treated by **7.120.**

**7.140** $1^\infty, 0^0, \infty^0$. If $\{F(x)\}^{(fx)}$ is indeterminate in any of these forms for $x = a$, its true value may be found by finding the true value of the logarithm of the given expression.

Example:

$$\left[\left(\frac{1}{x}\right)^{\tan x}\right]_{x\to 0}.$$

$$\left(\frac{1}{x}\right)^{\tan x} = y; \quad \log y = -\tan x \cdot \log x,$$

$$\left[\tan x \cdot \log x\right]_{x=0} = \left[\frac{\log x}{\cot x}\right]_{x=0} = \left[\frac{\frac{1}{x}}{\csc^2 x}\right]_{x=0} = \left[\frac{\sin x}{x} \cdot \sin x\right]_{x=0} = 0.$$

Hence,

$$\left[\left(\frac{1}{x}\right)^{\tan x}\right]_{x=0} = 1.$$

**7.141** If $f(x)$ and $x$ approach $\infty$ together, and $\frac{f(x+1)}{f(x)}$ approaches a definite limit, then,

$$\underset{x\to\infty}{\text{Limit}}\left[\{f(x)\}^{\frac{1}{x}}\right] = \underset{x\to\infty}{\text{Limit}}\ \frac{f(x+1)}{f(x)}.$$

**7.150** Differential Coefficients of the form $\frac{0}{0}$. In determining the differential coefficient $\frac{dy}{dx}$ from an equation $f(x, y) = 0$, by means of the formula,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{1}$$

it may happen that for a pair of values, $x = a$, $y = b$, satisfying $f(x, y) = 0$, $\frac{dy}{dx}$ takes the form $\frac{0}{0}$.

Writing $\frac{dy}{dx} = y'$, and applying **7.102** to the quotient (1), a quadratic equation is obtained for determining $y'$, giving, in general, two different determinate values. If $y'$ is still indeterminate, apply **7.102** again, giving a cubic equation for determining $y'$. This process may be continued until determinate values result.

Example:

$$f(x, y) = (x^2 + y^2)^2 - c^2xy = 0,$$

$$y' = -\frac{4x(x^2 + y^2) - c^2y}{4y(x^2 + y^2) - c^2x}.$$

For $x = 0$, $y = 0$, $y'$ takes the value $\frac{0}{0}$.

Applying **7.102**,

$$-y' = \frac{12x^2 + 4y^2 + (8xy - c^2)y'}{4y'(x^2 + 3y^2) + 8xy - c^2}.$$

Solving this quadratic equation in $y'$, the two determinate values, $y' = 0$, $y' = \infty$, result for $x = 0$, $y = 0$.

**7.17** Special Indeterminate Forms and Limiting Values. In the following the notation $[f(x)]_a$ means the limit approached by $f(x)$ as $x$ approaches $a$ as a limit.

**7.171**

1. $\left[\left(1+\frac{c}{x}\right)^x\right]_\infty = e^c$ ($c$ a constant).

2. $[\sqrt{x+c}-\sqrt{x}]_\infty = 0.$

3. $[\sqrt{x(x+c)}-x]_\infty = \frac{c}{2}.$

4. $[\sqrt{(x+c_1)(x+c_2)}-x]_\infty = \tfrac{1}{2}(c_1+c_2).$

5. $\left[\sqrt[n]{(x+c_1)(x+c_2)\ldots(x+c_n)}-x\right]_\infty = \frac{1}{n}(c_1+c_2+\ldots c_n).$

6. $\left[\frac{\log(c_1+c_2 e^x)}{x}\right]_\infty = 1.$

7. $\left[\log\left(c_1+c_2 e^x\right)\cdot\log\left(1+\frac{1}{x}\right)\right]_\infty = 1.$

8. $\left[\left(\frac{\log x}{x}\right)^{\frac{1}{x}}\right]_\infty = 1.$

9. $\left[\frac{x}{(\log x)^m}\right]_\infty = \infty.$

10. $\left[\frac{a^x}{x^m}\right]_\infty = \infty$ ($a>1$).

11. $\left[\frac{a^x}{x!}\right]_\infty = 0$ ($x$ a positive integer).

12. $\left[x^{\frac{1}{x}}\right]_\infty = 1.$

13. $\left[\frac{\log x}{x}\right]_\infty = 0.$

14. $\left[(a+bc^x)^{\frac{1}{x}}\right]_\infty = c$ ($c>1$).

15. $\left[\left(\frac{1}{a+be^x}\right)^{\frac{c}{x}}\right]_\infty = e^{-c}.$

16. $\left[\frac{x}{\alpha+\beta x^2}\cdot\log(a+be^x)\right]_\infty = \frac{1}{\beta}.$

17. $\left[\left(a+bx^m\right)^{\frac{1}{\alpha+\beta\log x}}\right]_\infty = e^{\frac{m}{\beta}}$ ($m>0$).

**7.172**

1. $\left[x \sin \frac{c}{x}\right]_\infty = c.$

2. $\left[x\left(1 - \cos \frac{c}{x}\right)\right]_\infty = 0.$

3. $\left[x^2\left(1 - \cos \frac{c}{x}\right)\right]_\infty = \frac{c^2}{2}.$

4. $\left[\left(\cos \frac{c}{x}\right)^x\right]_\infty = 1.$

5. $\left[\left(\cos \frac{c}{x}\right)^{x^2}\right]_\infty = e^{-\frac{c^2}{2}}.$

6. $\left[\left(\frac{\sin \frac{c}{x}}{\frac{c}{x}}\right)^x\right]_\infty = 1.$

7. $\left[\frac{\cot \frac{c}{x}}{x}\right]_\infty = \frac{1}{c}.$

8. $\left[\sin \frac{c}{x} \cdot \log (a + be^x)\right]_\infty = c.$

9. $\left[\left(\cos \sqrt{\frac{2c}{x}}\right)^x\right]_\infty = e^{-c}.$

10. $\left[\left(1 + a \tan \frac{c}{x}\right)^x\right]_\infty = e^{ac}.$

11. $\left[\left(\cos \frac{c}{x} + a \sin \frac{c}{x}\right)^x\right]_\infty = e^{ac}.$

**7.173**

1. $\left[\frac{\sin x}{x}\right]_0 = 1.$

2. $\left[\frac{\tan x}{x}\right]_0 = 1.$

3. $\left[\left(\frac{\sin nx}{x}\right)^m\right]_0 = n^m.$

4. $[\sin^{-1} x \cdot \cot x]_0 = 1.$

5. $\left[\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}^{\cot x}\right]_0 = e.$

**7.174**

1. $[x^x]_0 = 1.$

2. $\left[x^{\frac{1}{a + b \log x}}\right]_0 = e^{\frac{1}{b}}.$

3. $\left[x^{\frac{1}{\log (e^x - 1)}}\right]_0 = e.$

4. $\left[x^m \log \frac{1}{x}\right]_0 = 0 \qquad (m \geqslant 1).$

5. $[\log \cos x \cdot \cot x]_0 = 0.$

6. $\left[\log \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \cot x\right]_0 = 1.$

7. $\left[\frac{e^x - 1}{x}\right]_0 = 1.$

8. $[x^m \log x]_0 = 0 \qquad (m > 0).$

9. $\left[\frac{e^x - e^{-x} - 2x}{(e^x - 1)^3}\right]_0 = \frac{1}{3}.$

10. $[xe^{\frac{1}{x}}]_0 = \infty.$

11. $\left[\frac{e^x - e^{-x}}{\log (1 + x)}\right]_0 = 2.$

12. $\left[\frac{\log \tan 2x}{\log \tan x}\right]_0 = 1.$

**7.175**

1. $\left[x^{\frac{1}{1-x}}\right]_1 = \frac{1}{e}$.

2. $[(\pi - 2x)\tan x]_{\frac{\pi}{2}} = 2$.

3. $\left[\log\left(2 - \frac{x}{c}\right)\cdot \tan\frac{\pi x}{2c}\right]_c = \frac{2}{\pi}$.

4. $\left[(e^c - e^x)\tan\frac{\pi x}{2c}\right]_c = \frac{2c}{\pi}e^c$.

5. $\left[\cos^{-1}\frac{x}{c}\cdot\tan\frac{\pi x}{2c}\right]_c = \infty$

6. $[(a + be^{\tan x})^{\pi - 2x}]_{\frac{\pi}{2}} = e^2$.

7. $\left[\left(2 - \frac{2x}{\pi}\right)^{\tan x}\right]_{\frac{\pi}{2}} = e^{\frac{2}{\pi}}$

8. $[(\tan x)^{\tan 2x}]_{\frac{\pi}{4}} = \frac{1}{e}$.

**7.18** Limiting Values of Sums.

1. $\underset{n\to\infty}{\text{Limit}}\left(\frac{1^k + 2^k + 3^k + \ldots + n^k}{n^{k+1}}\right) = \frac{1}{k+1}$ if $k > -1$.
$\infty$ if $k < -1$.

2. $\underset{n\to\infty}{\text{Limit}}\left(\frac{1}{na} + \frac{1}{na+b} + \frac{1}{na+2b} + \ldots + \frac{1}{na+(n-1)b}\right)$
$= \frac{\log(a+b) - \log a}{b}$ $(a, b > 0)$.

3. $\underset{n\to\infty}{\text{Limit}}\left(\frac{n-1^2}{1\cdot 2\cdot(n+1)} + \frac{n-2^2}{2\cdot 3\cdot(n+2)} + \frac{n-3^2}{3\cdot 4\cdot(n+3)} + \ldots\right.$
$\left. + \frac{(n-n^2}{n\cdot(n+1)\cdot(n+n)}\right) = 1 - \log 2$.

4. $\underset{n\to\infty}{\text{Limit}}\left[\left(a + b\frac{\sqrt{1}}{n}\right)^2 + \left(a^2 + b\frac{\sqrt{2}}{n}\right)^2 + \left(a^3 + b\frac{\sqrt{3}}{n}\right)^2 + \ldots\right.$
$\left. + \left(a^n + b\frac{\sqrt{n}}{n}\right)^2\right] = \frac{a^2}{1-a^2} + \frac{b^2}{2}$,
if $a$ is a positive proper fraction.

5. $\underset{n\to\infty}{\text{Limit}}\left[\sqrt{a + \frac{b}{n}} + \sqrt{a^2 + \frac{b}{n}} + \sqrt{a^3 + \frac{b}{n}} + \ldots + \sqrt{a^n + \frac{b}{n}}\right] = \infty$,
if $b > 0$ and $a$ is a positive proper fraction.

6. $\underset{n\to\infty}{\text{Limit}}\left[\sqrt{a + \frac{b}{1\cdot n}} + \sqrt{a^2 + \frac{b}{2\cdot n}} + \sqrt{a^3 + \frac{b}{3\cdot n}} + \ldots + \sqrt{a^n + \frac{b}{n\cdot n}}\right]$
$= \frac{\sqrt{a}}{1-\sqrt{a}} + 2\sqrt{b}$,
if $b > 0$ and $a$ is a positive proper fraction.

7. $\underset{n\to\infty}{\text{Limit}}\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right] = \gamma = 0.5772157\ldots$
(6.602).

**7.19** Limiting Values of Products.

1. $$\underset{n\to\infty}{\text{Limit}}\left[\left(1+\frac{c}{n}\right)\left(1+\frac{c}{n+1}\right)\left(1+\frac{c}{n+2}\right)\cdots\left(1+\frac{c}{2n-1}\right)\right]=2^c,$$
if $c>0$.

2. $$\underset{n\to\infty}{\text{Limit}}\left[\left(1+\frac{c}{na}\right)\left(1+\frac{c}{na+b}\right)\left(1+\frac{c}{na+2b}\right)\cdots\left(1+\frac{c}{na+(n-1)b}\right)\right]=\left(1+\frac{b}{a}\right)^{\frac{c}{b}},$$
if $a$, $b$, $c$ are all positive.

3. $$\underset{n\to\infty}{\text{Limit}}\left[\frac{\{m(m+1)(m+2)\cdots\cdots(m+n-1)\}^{\frac{1}{n}}}{m+\frac{1}{2}(n-1)}\right]=\frac{2}{e},$$
if $m>0$.

4. $$\underset{n\to}{\text{Limit}}\left[\left(1+\frac{2c}{n^2}\right)\left(1+\frac{4c}{n^2}\right)\left(1+\frac{6c}{n^2}\right)\cdots\cdots\left(1+\frac{2nc}{n^2}\right)\right]=e^c.$$

**7.20** Maxima and Minima.

**7.201** Functions of One Variable. $y=f(x)$ is a maximum or minimum for the values of $x$ satisfying the equation, $f'(x)=\dfrac{\partial f(x)}{\partial x}=0$, provided that $f'(x)$ is continuous for these values of $x$.

**7.202** If, for $x=a$, $f'(a)=0$,

$y=f(a)$ is a maximum if $f''(a)<0$

$y=f(a)$ is a minimum if $f''(a)>0$.

Example:

$$y=\frac{x}{x^2+\alpha x+\beta},\qquad \beta>0,$$

$$f'(x)=\frac{-x^2+\beta}{(x^2+\alpha x+\beta)^2},$$

$$f'(x)=0 \text{ when } x=\pm\sqrt{\beta},$$

$$f''(x)=\frac{2x^3-6\beta x-2\alpha\beta}{(x^2+\alpha x+\beta)^3}$$

For $x=+\sqrt{\beta}$, $f''(x)=\dfrac{-2}{\sqrt{\beta}}\dfrac{1}{(2\sqrt{\beta}+\alpha)^2}$ Maximum,

$$\text{For } x = -\sqrt{\beta},\ f''(x) = \frac{+2}{\sqrt{\beta}} \frac{1}{(2\sqrt{\beta} - \alpha)^2} \quad \text{Minimum},$$

$$y_{max} = \frac{1}{\alpha + 2\sqrt{\beta}},$$

$$y_{min} = \frac{1}{\alpha - 2\sqrt{\beta}}.$$

**7.203** If for $x = a$, $f'(a) = 0$ and $f''(a) = 0$, in order to determine whether $y = f(a)$ is a maximum or minimum it is necessary to form the higher differential coefficients, until one of even order is found which does not vanish for $x = a$. $y = f(a)$ is a maximum or minimum according as the first of the differential coefficients, $f''(a)$, $f^{iv}(a)$, $f^{vi}(a)$, . . . . . of even order which does not vanish is negative or positive.

**7.210** Functions of Two Variables. $F(x, y)$ is a maximum or minimum for the pair of values of $x$ and $y$ that satisfy the equations,

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0,$$

and for which

$$\left(\frac{\partial^2 F}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} < 0.$$

If both $\frac{\partial^2 F}{\partial x^2}$ and $\frac{\partial^2 F}{\partial y^2}$ are negative for this pair of values of $x$ and $y$, $F(x, y)$ is a maximum. If they are both positive $F(x, y)$ is a minimum.

**7.220** Functions of $n$ Variables. For the maximum or minimum of a function of $n$ variables, $F(x_1, x_2 \ldots\ldots, x_n)$, it is necessary that the first derivatives, $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots\ldots, \frac{\partial F}{\partial x_n}$ all vanish; and that the lowest order of the higher derivatives which do not all vanish be an even number. If this number be 2 the necessary condition for a minimum is that all of the determinants,

$$D_k = \begin{vmatrix} f_{11} & f_{12} & \ldots\ldots & f_{1k} \\ f_{21} & f_{22} & \ldots\ldots & f_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ f_{k1} & f_{k2} & \ldots\ldots & f_{kk} \end{vmatrix}, \quad k = 1, 2, \ldots . n,$$

where

$$f_{ij} = \frac{\partial^2 F}{\partial x_i\, \partial x_j},$$

shall be positive. For a maximum the determinants must be alternately negative and positive, beginning with $D_1 = \frac{\partial^2 F}{\partial x_1^2}$ negative.

**7.230** Maxima and Minima with Conditions. If $F(x_1, x_2, \ldots\ldots, x_n)$ is to be made a maximum or minimum subject to the conditions,

$$1.\quad \begin{cases} \phi_1(x_1, x_2, \ldots\ldots, x_n) = 0 \\ \phi_2(x_1, x_2, \ldots\ldots, x_n) = 0 \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \phi_k(x_1, x_2, \ldots\ldots, x_n) = 0, \end{cases}$$

where $k < n$, the necessary conditions are,

$$2.\qquad \frac{\partial F}{\partial x_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial \phi_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots . n,$$

where the $\lambda$'s are $k$ undetermined multipliers. The $n$ equations (2) together with the $k$ equations of condition (1) furnish $k + n$ equations to determine the $k + n$ quantities, $x_1, x_2, \ldots\ldots, x_n, \lambda_1, \lambda_2, \ldots\ldots, \lambda_k$.

Example:

To find the axes of the ellipsoid, referred to its center as origin,

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz = 1.$$

Denoting the radius vector to the surface by $r$, and its direction-cosines by $l, m, n$, so that $x = lr$, $y = mr$, $z = nr$, it is necessary to find the maxima and minima of

$$r^2 = \frac{1}{a_{11}l^2 + a_{22}m^2 + a_{33}n^2 + 2a_{12}lm + 2a_{23}m + 2a_{13}lnn'}$$

subject to the condition

$$\phi(l, m, n) = l^2 + m^2 + n^2 - 1 = 0.$$

This is the same as finding the minima and maxima of

$$F(l, m, n) = a_{11}l^2 + a_{22}m^2 + a_{33}n^2 + 2a_{12}lm + 2a_{23}mn + 2a_{13}ln.$$

Equation (2) gives:

$$\begin{aligned} (a_{11} + \lambda)l + a_{12}m + a_{13}n &= 0, \\ a_{12}l + (a_{22} + \lambda)m + a_{23}n &= 0, \\ a_{13}l + a_{23}m + (a_{33} + \lambda)n &= 0. \end{aligned}$$

Multiplying these 3 equations by $l, m, n$ respectively and adding,

$$\lambda = -\frac{1}{r^2}.$$

Then by (1. **1.363**) the 3 values of $r$ are given by the 3 roots of

$$\begin{vmatrix} a_{11} - \frac{1}{r^2} & a_{12} & a_{13} \\ a_{12} & a_{22} - \frac{1}{r^2} & a_{23} \\ a_{13} & a_{23} & a_{33} - \frac{1}{r^2} \end{vmatrix} = 0.$$

## 7.30 Derivatives.

## 7.31 First Derivatives.

1. $\frac{dx^n}{dx^n} = nx^{n-1}$.

2. $\frac{da^x}{dx} = a^x \log a$.

3. $\frac{de^x}{dx} = e^x$.

4. $\frac{dx^x}{dx} = x^x(1 + \log x)$.

5. $\frac{d \log_a x}{dx} = \frac{1}{x \log a} = \frac{\log_a e}{x}$.

6. $\frac{d \log x}{dx} = \frac{1}{x}$.

7. $\frac{dx^{\log x}}{dx} = 2x^{\log x - 1} \log x$.

8. $\frac{d(\log x)^x}{dx} = (\log x)^{x-1} \{1 + \log x \cdot \log \log x\}$.

9. $\frac{d\left(\frac{x}{e}\right)^x}{dx} = \left(\frac{x}{e}\right)^x \log x$.

10. $\frac{d \sin x}{dx} = \cos x$.

11. $\frac{d \cos x}{dx} = -\sin x$.

12. $\frac{d \tan x}{dx} = \sec^2 x$.

13. $\frac{d \cot x}{dx} = -\csc^2 x$.

14. $\frac{d \sec x}{dx} = \sec^2 x \cdot \sin x$.

15. $\frac{d \csc x}{dx} = -\csc^2 x \cdot \cos x$.

16. $\frac{d \sin^{-1} x}{dx} = -\frac{d \cos^{-1} x}{dx} = \frac{1}{\sqrt{1 - x^2}}$.

17. $\frac{d \tan^{-1} x}{dx} = -\frac{d \cot^{-1} x}{dx} = \frac{1}{1 + x^2}$.

18. $\frac{d \sec^{-1} x}{dx} = -\frac{d \csc^{-1} x}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$.

19. $\frac{d \sinh x}{dx} = \cosh x$.

20. $\frac{d \cosh x}{dx} = \sinh x$.

21. $\frac{d \tanh x}{dx} = \operatorname{sech}^2 x.$

22. $\frac{d \coth x}{dx} = -\operatorname{csch}^2 x.$

23. $\frac{d \operatorname{sech} x}{dx} = -\operatorname{sech} x \cdot \tanh x.$

24. $\frac{d \operatorname{csch} x}{dx} = -\operatorname{csch} x \cdot \coth x.$

25. $\frac{d \sinh^{-1} x}{dx} = \frac{1}{\sqrt{x^2+1}}.$

26. $\frac{d \cosh^{-1} x}{dx} = \frac{1}{\sqrt{x^2-1}}.$

27. $\frac{d \tanh^{-1} x}{dx} = \frac{d \coth^{-1} x}{dx} = \frac{1}{1-x^2}.$

28. $\frac{d \operatorname{sech}^{-1} x}{dx} = -\frac{1}{x\sqrt{1-x^2}}.$

29. $\frac{d \operatorname{csch}^{-1} x}{dx} = -\frac{1}{x\sqrt{1+x^2}}.$

30. $\frac{d\, gd\, x}{dx} = \operatorname{sech} x.$

31. $\frac{d\, gd^{-1} x}{dx} = \sec x.$

**7.32**

1. $\frac{d(y_1 y_2 y_3 \ldots y_n)}{dx} = y_1 y_2 \ldots y_n \left(\frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} + \ldots + \frac{1}{y_n}\frac{dy_n}{dx}\right).$

2. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

3. $\frac{da^u}{dx} = a^u \frac{du}{dx} \log a.$

4. $\frac{de^u}{dx} = e^u \frac{du}{dx}.$

5. $\frac{df(u)}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}.$

**7.33** Derivative of a Definite Integral.

1. $\frac{d}{da}\int_{\psi(a)}^{\phi(a)} f(x,a)dx = f(\phi(a),a)\frac{d\phi(a)}{da} - f(\psi(a),a)\frac{d\psi(a)}{da} + \int_{\psi(a)}^{\phi(a)} \frac{d}{da} f(x,a)dx.$

2. $\frac{d}{da}\int_b^a f(x)dx = f(a).$

3. $\frac{d}{db}\int_b^a f(x)dx = -f(b).$

**7.35** Higher Derivatives.

**7.351** Leibnitz's Theorem. If $u$ and $v$ are functions of $x$,

$$\frac{d^n(uv)}{dx^n} = u\frac{d^nv}{dx^n} + \frac{n}{1!}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{2!}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}}$$
$$+ \frac{n(n-1)(n-2)}{3!}\frac{d^3u}{dx^3}\frac{d^{n-3}v}{dx^{n-3}} + \ldots\ldots + v\frac{d^nu}{dx^n}.$$

**7.352** Symbolically,

$$\frac{d^n(uv)}{dx^n} = (u+v)^{(n)},$$

where

$$u^0 = u, \quad v^0 = v.$$

**7.353**
$$\frac{d^ne^{ax}u}{dx^n} = e^{ax}\left(a + \frac{d}{dx}\right)^n u.$$

**7.354** If $\phi\left(\frac{d}{dx}\right)$ is a polynomial in $\frac{d}{dx}$,

$$\phi\left(\frac{d}{dx}\right)e^{ax}u = e^{ax}\phi\left(a + \frac{d}{dx}\right)u.$$

**7.355** Euler's Theorem. If $u$ is a homogeneous function of the $n$th degree of $r$ variables, $x_1, x_2, \ldots x_r$,

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \ldots\ldots + x_r\frac{\partial}{\partial x_r}\right)^m u = n^m u,$$

where $m$ may be any integer, including 0.

## 7.36 Derivatives of Functions of Functions.

**7.361** If $f(x) = F(y)$, and $y = \phi(x)$,

1. $$\frac{d^n}{dx^n}f(x) = \frac{U_1}{1!}F'(y) + \frac{U_2}{2!}F''(y) + \frac{U_3}{3!}F'''(y) + \ldots\ldots + \frac{U_n}{n!}F^{(n)}(y),$$

where

2. $$U_k = \frac{\partial^n}{\partial x^n}y^k - \frac{k}{1!}y\frac{\partial^n}{\partial x^n}y^{k-1} + \frac{k(k-1)}{2!}y^2\frac{\partial^n}{\partial x^n}y^{k-2} - \ldots\ldots$$

**7.362**

1. $$(-1)^n\frac{d^n}{dx^n}F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}}F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}}\frac{n}{1!}F^{(n-1)}\left(\frac{1}{x}\right)$$
$$+ \frac{(n-1)(n-2)}{x^{2n-2}}\cdot\frac{n(n-1)}{2!}F^{(n-2)}\left(\frac{1}{x}\right) + \ldots\ldots$$

2. $$(-1)^n\frac{d^n}{dx^n}e^{\frac{a}{x}} = \frac{1}{x^n}e^{\frac{a}{x}}\left\{\left(\frac{a}{x}\right)^n + (n-1)\frac{n}{1!}\left(\frac{a}{x}\right)^{n-1}\right.$$
$$+ (n-1)(n-2)\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^{n-2}$$
$$\left. + (n-1)(n-2)(n-3)\frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^{n-3} + \ldots\right\}.$$

**7.363**

1. $$\frac{d^n}{dx^n} F(x^2) = (2x)^n F^{(n)}(x^2) + \frac{n(n-1)}{1!}(2x)^{n-2} F^{(n-1)}(x^2) + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} F^{(n-2)}(x^2) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!}(2x)^{n-6} F^{(n-3)}(x^2) + \dots$$

2. $$\frac{d^n}{dx^n} e^{ax^2} = (2ax)^n e^{ax^2}\left\{1 + \frac{n(n-1)}{1!(4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2!(4ax^2)^2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!(4ax^2)^3} + \dots\right\}.$$

3. $$\frac{d^n}{dx^n}(1+ax^2)^\mu = \frac{\mu(\mu-1)(\mu-2)\dots(\mu-n+1)(2ax)^n}{(1+ax^2)^{n-\mu}}\left\{1 + \frac{n(n-1)}{1\cdot(\mu-n+1)}\frac{(1+ax^2)}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(\mu-n+1)(\mu-n+2)}\left(\frac{1+ax^2}{4ax^2}\right)^2 + \dots\right\}.$$

4. $$\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{m-\frac{1}{2}} = (-1)^{m-1}\frac{1\cdot3\cdot5\dots(2m-1)}{m}\sin(m\cos^{-1}x).$$

**7.364**

1. $$\frac{d^n}{dx^n} F(\sqrt{x}) = \frac{F^{(n)}(\sqrt{x})}{(2\sqrt{x})^n} - \frac{n(n-1)}{1!}\frac{F^{(n-1)}(\sqrt{x})}{(2\sqrt{x})^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!}\frac{F^{(n-2)}(\sqrt{x})}{(2\sqrt{x})^{n+2}} - \dots$$

2. $$\frac{d^n}{dx^n}(1+a\sqrt{x})^{2n-1} = \frac{1\cdot3\cdot5\dots(2n-1)}{2^n}\frac{a}{\sqrt{x}}\left(a^2 - \frac{1}{x}\right)^{n-1}.$$

**7.365**

1. $$\frac{d^n}{dx^n} F(e^x) = \frac{E_1}{1!}e^x F'(e^x) + \frac{E_2}{2!}e^{2x}F''(e^x) + \frac{E_3}{3!}e^{3x}F'''(e^x) + \dots$$

where

2. $$E_k = k^n - \frac{k}{1!}(k-1)^n + \frac{k(k-1)}{2!}(k-2)^n - \dots.$$

3. $$\frac{d^n}{dx^n}\frac{1}{1+e^{2x}} = -E_1 e^x \frac{\sin(2\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^2}} + E_2 e^{2x}\frac{\sin(3\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^3}} - E_3 e^{3x}\frac{\sin(4\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^4}} + \dots.$$

4. $$\frac{d^n}{dx^n}\frac{e^x}{1+e^{2x}} = -E_1 e^x \frac{\cos(2\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^2}} + E_2 e^{2x}\frac{\cos(3\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^3}} - E_3 e^{3x}\frac{\cos(4\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^4}} + \dots.$$

**7.366**

1. $$\frac{d^n}{dx^n} F(\log x) = \frac{1}{x^n}\left\{ \overset{n}{C}_0 F^{(n)}(\log x) - \overset{n}{C}_1 F^{(n-1)}(\log x) + \overset{n}{C}_2 F^{(n-2)}(\log x) - \dots \right\}.$$

$$\overset{n}{C}_0 = 1,$$

$$\overset{n}{C}_1 = 1 + 2 + 3 + \dots + (n-1) = \frac{n(n-1)}{2},$$

$$\begin{aligned}\overset{n}{C}_2 = {} & 1\cdot 2 + 1\cdot 3 + 1\cdot 4 + \dots + 1\cdot(n-1) \\ & + 2\cdot 3 + 2\cdot 4 + \dots + 2\cdot(n-1) \\ & + 3\cdot 4 + \dots + 3\cdot(n-1) \\ & + \dots \\ & + (n-2)(n-1) = \frac{n(n-1)(n-2)(3n-1)}{24}.\end{aligned}$$

2. $$\overset{n+1}{C}_k = \overset{n}{C}_k + n\overset{n}{C}_{k-1}.$$

3. $$\overset{-n}{C}_k = \overset{-(n-1)}{C}_k + n\overset{-n}{C}_{k-1}.$$

$$\overset{n}{C}_0 = 1 \quad \overset{k}{C}_k = 0,$$

$$\overset{2}{C}_1 = 1 \quad \overset{3}{C}_1 = 3 \quad \overset{4}{C}_1 = 6,$$

$$\overset{3}{C}_2 = 2 \quad \overset{4}{C}_2 = 11,$$

$$\overset{4}{C}_3 = 6.$$

$$\overset{-n}{C}_0 = 1 \quad \overset{-1}{C}_k = 1,$$

$$\overset{-2}{C}_1 = 3 \quad \overset{-3}{C}_1 = 6 \quad \overset{-4}{C}_1 = 10,$$

$$\overset{-2}{C}_2 = 7 \quad \overset{-3}{C}_2 = 25 \quad \overset{-4}{C}_2 = 65,$$

$$\overset{-2}{C}_3 = 15 \quad \overset{-3}{C}_3 = 90 \quad \overset{-4}{C}_3 = 350.$$

**7.367** Table of $\overset{n}{C}_k$.

| $n =$ | $-4$ | $-3$ | $-2$ | $-1$ | $+1$ | $+2$ | $+3$ | $+4$ | $+5$ | $+6$ | $+7$ | $+8$ | $+9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_0 =$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $C_1 =$ | 10 | 6 | 3 | 1 | ... | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 |
| $C_2 =$ | 65 | 25 | 7 | 1 | ... | .... | 2 | 11 | 35 | 85 | 175 | 322 | 546 |
| $C_3 =$ | 350 | 90 | 15 | 1 | ... | .... | .... | 6 | 50 | 225 | 735 | 1960 | 4536 |
| $C_4 =$ | 1701 | 301 | 31 | 1 | ... | .... | .... | .... | 24 | 274 | 1624 | 6769 | 22449 |
| $C_5 =$ | 7770 | 966 | 63 | 1 | ... | .... | .... | .... | .... | 120 | 1764 | 13132 | 67284 |
| $C_6 =$ | 34105 | 3025 | 127 | 1 | ... | .... | .... | .... | .... | .... | 720 | 13068 | 118124 |
| $C_7 =$ | 145750 | 9330 | 225 | 1 | ... | .... | .... | .... | .... | .... | .... | 5040 | 109584 |
| $C_8 =$ | 611501 | 28501 | 511 | 1 | ... | .... | .... | .... | .... | .... | .... | ..... | 40320 |

**7.368**

1. $$\frac{d^n}{dx^n}(\log x)^p = \frac{(-1)^{n-1}}{x^n}\left\{ {}^{n}C_{n-1}p(\log x)^{p-1} - {}^{n}C_{n-2}p(p-1)(\log x)^{p-2} + {}^{n}C_{n-3}p(p-1)(p-2)(\log x)^{p-3} - \ldots \right\},$$

where $p$ is a positive integer. If $n<p$ there are $n$ terms in the series. If $n \geqslant p$,

2. $$\frac{d^n}{dx^n}(\log x)^p = \frac{(-1)^{n-1}}{x^n}\left\{ {}^{n}C_{n-1}p(\log x)^{p-1} - {}^{n}C_{n-2}p(p-1)(\log x)^{p-2} + \ldots\ldots + (-1)^{p+1}\,{}^{n}C_{n-p}p(p-1)(p-2)\ldots\ldots 2\cdot 1 \right\}.$$

**7.369** $$\left\{\log(1+x)\right\}^p = {}^{p}C_0 x^p - {}^{p+1}C_1\frac{x^{p+1}}{p+1} + {}^{p+2}C_2\frac{x^{p+2}}{(p+1)(p+2)} - \ldots\ldots$$
$$-1 < x < +1.$$

**7.37** Derivatives of Powers of Functions. If $y = \phi(x)$.

1. $$\frac{d^n}{dx^n}y^p = p\binom{n-p}{n}\left\{ -\binom{n}{1}\frac{1}{p-1}y^{p-1}\frac{d^n y}{dx^n} + \binom{n}{2}\frac{1}{p-2}y^{p-2}\frac{d^n y^2}{dx^n} - \ldots\ldots \right\}.$$

2. $$\frac{d^n}{dx^n}\log y = \binom{n}{1}\frac{1}{1\cdot y}\frac{d^n y}{dx^n} - \binom{n}{2}\frac{1}{2\cdot y^2}\frac{d^n y^2}{dx^n} + \binom{n}{3}\frac{1}{3\cdot y^3}\frac{d^n y^3}{dx^n} - \ldots\ldots$$

**7.38**

1. $$\frac{d^n(a+bx)^m}{dx^n} = m(m-1)(m-2)\ldots\ldots(m-[n-1])\,b^n(a+bx)^{m-n}.$$

2. $$\frac{d^n(a+bx)^{-1}}{dx^n} = (-1)^n\frac{n!b^n}{(a+bx)^{n+1}}.$$

3. $$\frac{d^n(a+bx)^{-\frac{1}{2}}}{dx^n} = (-1)^n\frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)}{2^n(a+bx)^{n+\frac{1}{2}}}b^n.$$

4. $$\frac{d^n\log(a+bx)}{dx^n} = (-1)^{n-1}\frac{(n-1)!b^n}{(a+bx)^n}.$$

5. $$\frac{d^n e^{ax}}{dx^n} = a^n e^{ax}.$$

6. $$\frac{d^n\sin x}{dx^n} = \sin(\tfrac{1}{2}n\pi + x).$$

7. $$\frac{d^n\cos x}{dx^n} = \cos(\tfrac{1}{2}n\pi + x).$$

8. $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = (-1)^n \dfrac{n!}{x^{n+1}} \left\{ \log x - \left(\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\right)\right\}.$

9. $\dfrac{d^{n+1}}{dx^{n+1}} \sin^{-1}x = \dfrac{1\cdot 3\cdot 5 \ldots (2n-1)}{2^n(1-x)^n\sqrt{1-x^2}} \left\{ 1 - \dfrac{1}{2n-1}\binom{n}{1}\dfrac{1-x}{1+x}\right.$

$$+ \frac{1\cdot 3}{(2n-1)(2n-3)}\binom{n}{2}\left(\frac{1-x}{1+x}\right)^2 - \frac{1\cdot 3\cdot 5}{(2n-1)(2n-3)(2n-5)}\binom{n}{3}\left(\frac{1-x}{1+x}\right)^3$$

$$\left. + \ldots\ldots\right\}.$$

10. $\dfrac{d^n}{dx^n}(\tan^{-1}x) = (-1)^{n-1}\dfrac{(n-1)!}{(1+x^2)^{\frac{n}{2}}} \sin\left(n \tan^{-1}\dfrac{1}{x}\right).$

**7.39** Derivatives of Implicit Functions.

**7.391** If $y$ is a function of $x$, and $f(x, y) = 0$.

1. $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial}{\partial x}}{\dfrac{\partial f}{\partial y}}.$

2. $\dfrac{d^2y}{dx^2} = -\dfrac{\left(\dfrac{\partial f}{\partial y}\right)^2\dfrac{\partial^2 f}{\partial x^2} - 2\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial x\partial y} + \left(\dfrac{\partial f}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial y^2}}{\left(\dfrac{\partial f}{\partial y}\right)^3}$

**7.392** If $z$ is a function of $x$ and $y$, and $f(x, y, z) = 0$.

1. $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}; \quad \dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}}.$

2. $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{\left(\dfrac{\partial f}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial x^2} - 2\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial x}\dfrac{d^2 f}{\partial x\partial z} + \left(\dfrac{\partial f}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial z}\right)^3}.$

3. $\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{\left(\dfrac{\partial f}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial y^2} - 2\dfrac{\partial f}{\partial z}\dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial y\partial z} + \left(\dfrac{\partial f}{\partial y}\right)^2\dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial z}\right)^3}.$

4. $\dfrac{\partial^2 z}{\partial x\partial y} = -\dfrac{\left(\dfrac{\partial f}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial x\partial y} - \dfrac{\partial f}{\partial z}\left(\dfrac{\partial f}{\partial x}\dfrac{\partial^2 f}{\partial y\partial z} + \dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial x\partial z}\right) + \dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial z}\right)^3}.$

# VIII. DIFFERENTIAL EQUATIONS.

**8.000** Ordinary differential equations of the first order. General form:

$$\frac{dy}{dx} = f(x, y).$$

**8.001** Variables are separable. $f(x, y)$ is of, or can be reduced to, the form:

$$f(x, y) = -\frac{X}{Y},$$

where $X$ is a function of $x$ alone and $Y$ is a function of $y$ alone.
The solution is:

$$\int X\,dx + \int Y\,dy = C.$$

**8.002** Linear equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Solution:

$$y = e^{-\int P(x)dx}\left\{\int Q(x)e^{-\int P(x)dx}\,dx + C\right\}.$$

**8.003** Equations of the form:

$$\frac{dy}{dx} + P(x)y = y^n Q(x).$$

Solution:

$$\frac{1}{y^{n-1}}e^{-(n-1)\int P(x)dx} + (n-1)\int Q(x)e^{-(n-1)\int P(x)dx}dx = C.$$

**8.010** Homogeneous equations of the form:

$$\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)},$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous functions of $x$ and $y$ of the same degree. The change of variable:

$$y = vx,$$

gives the solution:

$$\int \frac{dv}{\dfrac{P(1, v)}{Q(1, v)} + v} + \log x = C.$$

**8.011** Equations of the form:

$$\frac{dy}{dx} = \frac{a'x + b'y + c'}{ax + by + c}.$$

If $ab' - a'b \neq 0$, the substitution

$$x = x' + p, \quad y = y' + q,$$

where

$$ap + bq + c = 0,$$
$$a'p + b'q + c' = 0,$$

renders the equation homogeneous, and it may be solved by **8.010.**

If $ab' - a'b = 0$ and $b' \neq 0$, the change of variables to either $x$ and $z$ or $y$ and $z$ by means of

$$z = ax + by,$$

will make the variables separable (**8.001**).

**8.020** Exact differential equations. The equation,

$$P(x, y)dx + Q(x, y)dy = 0,$$

is exact if,

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

The solution is:

$$\int P(x, y)dx + \int \left\{ Q(x, y) - \frac{\partial}{\partial y} \int P(x, y)dx \right\} dy = C,$$

or

$$\int Q(x, y)dy + \int \left\{ P(x, y) - \frac{\partial}{\partial x} \int Q(x, y)dy \right\} dx = C.$$

**8.030** Integrating factors. $v(x, y)$ is an integrating factor of

$$P(x, y)\ dx + Q(x, y)\ dy = 0,$$

if

$$\frac{\partial}{\partial x}(vQ) = \frac{\partial}{\partial y}(vP).$$

**8.031** If one only of the functions $Px + Qy$ and $Px - Qy$ is equal to 0, the reciprocal of the other is an integrating factor of the differential equation.

**8.032** Homogeneous equations. If neither $Px + Qy$ nor $Px - Qy$ is equal to 0,

$\frac{1}{Px + Qy}$ is an integrating factor of the equation if it is homogeneous.

**8.033** An equation of the form,

$$P(x, y)y\,dx + Q(x, y)x\,dy = 0,$$

has an integrating factor:

$$\frac{1}{xP - yQ}.$$

**8.034** If

$$\frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Q} = F(x)$$

is a function of $x$ only, an integrating factor is

$$e^{\int F(x)dx}.$$

**8.035** If

$$\frac{\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}}{P} = F(y)$$

is a function of $y$ only, an integrating factor is

$$e^{\int F(y)dy}.$$

**8.036** If

$$\frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Qy - Px} = F(xy)$$

is a function of the product $xy$ only, an integrating factor is

$$e^{\int F(xy)d(xy)}.$$

**8.037** If

$$\frac{x^2\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)}{Px + Qy} = F\left(\frac{y}{x}\right)$$

is a function of the quotient $\frac{y}{x}$ only, an integrating factor is

$$e^{\int F\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)}.$$

**8.040** Ordinary differential equations of the first order and of degree higher than the first.

Write:

$$\frac{dy}{dx} = p.$$

General form of equation:

$$f(x, y, p) = 0.$$

**8.041** The equation can be solved as an algebraic equation in $p$. It can be written

$$(p - R_1)(p - R_2) \ldots\ldots\ldots (p - R_n) = 0.$$

The differential equations:

$$p = R_1(x, y),$$
$$p = R_2(x, y),$$
$$\ldots\ldots$$

may be solved by the previous methods. Write the solutions:

$$f_1(x, y, c) = 0; \quad f_2(x, y, c) = 0; \quad \ldots\ldots\ldots$$

where $c$ is the same arbitrary constant in each. The solution of the given differential equation is:

$$f_1(x, y, c) f_2(x, y, c) \ldots\ldots\ldots\ldots f_n(x, y, c) = 0.$$

**8.042** The equation can be solved for $y$:

1. $$y = f(x, p).$$

Differentiate with respect to $x$:

2. $$p = \psi\left(x, p, \frac{dp}{dx}\right).$$

It may be possible to integrate (2) regarded as an equation in the two variables $x$, $p$, giving a solution

3. $$\phi(x, p, c) = 0.$$

If $p$ is eliminated between (1) and (3) the result will be the solution of the given equation.

**8.043** The equation can be solved for $x$:

1. $$x = f(y, p).$$

Differentiate with respect to $y$:

2. $$\frac{1}{p} = \psi\left(y, p, \frac{dp}{dy}\right).$$

If a solution of (2) can be found:

3. $$\phi\ (y, p, c) = 0.$$

Eliminate $p$ between (1) and (3) and the result will be the solution of the given equation.

**8.044** The equation does not contain $x$:

$$f(y, p) = 0.$$

It may be solved for $p$, giving,

$$\frac{dy}{dx} = F(y),$$

which can be integrated.

**8.045** The equation does not contain $y$:

$$f(x, p) = 0.$$

It may be solved for $p$, giving,

$$\frac{dy}{dx} = F(x),$$

which can be integrated.
It may be solved for $x$, giving,

$$x = F(p),$$

which may be solved by **8.043.**

**8.050** Equations homogeneous in $x$ and $y$.
General form:

$$F\left(p, \frac{y}{x}\right) = 0.$$

(a) Solve for $p$ and proceed as in **8.001**

(b) Solve for $\frac{y}{x}$:

$$y = xf(p).$$

Differentiate with respect to $x$:

$$\frac{dx}{x} = \frac{f'(p)dp}{p - f(p)},$$

which may be integrated.

**8.060** Clairaut's differential equation:

1. $$y = px + f(p),$$

the solution is:

$$y = cx + f(c).$$

The singular solution is obtained by eliminating $p$ between (1) and

2. $$x + f'(p) = 0.$$

**8.061** The equation

1. $$y = xf(p) + \phi(p).$$

The solution is that of the linear equation of the first order:

2. $$\frac{dx}{dp} - \frac{f'(p)}{p - f(p)}\, x = \frac{\phi'(p)}{p - f(p)},$$

which may be solved by **8.002.** Eliminating $p$ between (1) and the solution of (2) gives the solution of the given equation.

**8.062** The equation:

$$x\phi(p) + y\psi(p) = \chi(p),$$

may be reduced to **8.061** by dividing by $\psi(p)$.

## DIFFERENTIAL EQUATIONS OF AN ORDER HIGHER THAN THE FIRST

**8.100** Linear equations with constant coefficients. General form:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = V(x).$$

The complete solution consists of the sum of

(a) The complementary function, obtained by solving the equation with $V(x) = 0$, and containing $n$ arbitrary constants, and

(b) The particular integral, with no arbitrary constants.

**8.101** The complementary function. Assume $y = e^{\lambda x}$. The equation for determining $\lambda$ is:

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0.$$

**8.102** If the roots of **8.101** are all real and distinct the complementary function is:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \ldots + c_n e^{\lambda_n x}.$$

**8.103** For a pair of complex roots:

$$\mu \pm i\nu,$$

the corresponding terms in the complementary function are:

$$e^{\mu x}(A \cos \nu x + B \cos \nu x) = Ce^{\mu x} \cos (\nu x - \theta) = Ce^{\mu x} \sin (\nu x + \theta),$$

where

$$C = \sqrt{A^2 + B^2}, \quad \tan \theta = \frac{B}{A}.$$

**8.104** If there are $r$ equal real roots the terms in the complementary function corresponding to them are:

$$e^{\lambda x}(A_1 + A_2 x + A_3 x^2 + \ldots + A_r x^{r-1}),$$

where $\lambda$ is the repeated root, and $A_1, A_2, \ldots, A_r$ are the $r$ arbitrary constants.

**8.105** If there are $m$ equal pairs of complex roots the terms in the complementary function corresponding to them are:

$$\begin{aligned} &e^{\mu x}\{(A_1 + A_2 x + A_3 x^2 + \ldots + A_m x^{m-1}) \cos \nu x \\ &\qquad + (B_1 + B_2 x + B_3 x^2 + \ldots + B_m x^{m-1}) \sin \nu x\} \\ &= e^{\mu x}\{C_1 \cos (\nu x - \theta_1) + C_2 x \cos (\nu x - \theta_2) + \ldots + C_m x^{m-1} \cos (\nu x - \theta_m)\} \\ &= e^{\mu x}\{C_1 \sin (\nu x + \theta_1) + C_2 x \sin (\nu x + \theta_2) + \ldots + C_m x^{m-1} \sin (\nu x + \theta_m)\} \end{aligned}$$

where $\lambda \pm i\mu$ is the repeated root and

$$C_k = \sqrt{A_k^2 + B_k^2},$$

$$\tan \theta_k = \frac{B_k}{A_k}.$$

The particular integral.

**8.110** The operator $D$ stands for $\frac{\partial}{\partial x}$, $D^2$ for $\frac{\partial^2}{\partial x^2}$, . . . . . .

The differential equation **8.100** may be written:

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)y = f(D)y = V(x)$$

$$y = \frac{V(x)}{f(D)},$$

$$f(D) = (D - \lambda_1)(D - \lambda_2) \ldots\ldots (D - \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are determined as in **8.101.** The particular integral is:

$$y = e^{\lambda_1 x} \int e^{(\lambda_2 - \lambda_1)x}\, dx \int e^{(\lambda_3 - \lambda_2)x}\, dx \ldots\ldots \int e^{-\lambda_n(x)}\, V(x)dx.$$

**8.111** $\frac{1}{f(D)}$ may be resolved into partial fractions:

$$\frac{1}{f(D)} = \frac{N_1}{D - \lambda_1} + \frac{N_2}{D - \lambda_2} + \ldots\ldots + \frac{N_n}{D - \lambda_n}.$$

The particular integral is:

$$y = N_1 e^{\lambda_1 x} \int e^{-\lambda_1 x} V(x)dx + N_2 e^{\lambda_2 x} \int e^{-\lambda_2 x} V(x)dx + \ldots\ldots$$
$$+ N_n e^{\lambda_n x} \int e^{-\lambda_n x} V(x)dx.$$

THE PARTICULAR INTEGRAL IN SPECIAL CASES

**8.120** $V(x) = \text{const.} = c,$

$$y = \frac{c}{a_n}.$$

**8.121** $V(x)$ is a rational integral function of $x$ of the $m$th degree. Expand $\frac{1}{f(D)}$ in ascending powers of $D$, ending with $D^m$. Apply the operators $D$, $D^2$, . . . . ., $D^m$ to each term of $V(x)$ separately and the particular integral will be the sum of the results of these operations.

**8.122** $$V(x) = ce^{kx},$$

$$y = \frac{c}{f(k)} e^{kx},$$

unless $k$ is a root of $f(D) = 0$. If $k$ is a multiple root of order $r$ of $f(D) = 0$

$$y = \frac{cx^r e^{kx}}{r!\psi(k)},$$

where

$$f(D) = (D - k)^r \psi(D).$$

**8.123** $$V(x) = c \cos (kx + \alpha).$$

If $ik$ is not a root of $f(D) = 0$ the particular integral is the real part of

$$\frac{c}{f(ik)} e^{i(kx+\alpha)}.$$

If $ik$ is a multiple root of order $r$ of $f(D) = 0$ the particular integral is the real part of

$$\frac{cx^r e^{i(kx+\alpha)}}{f^{(r)}(ik)},$$

where $f^{(r)}(ik)$ is obtained by taking the $r$th derivative of $f(D)$ with respect to $D$, and substituting $ik$ for $D$.

**8.124** $$V(x) = c \sin (kx + \alpha).$$

If $ik$ is not a root of $f(D) = 0$ the particular integral is the real part of

$$\frac{-ic e^{i(kx+\alpha)}}{f(ik)}.$$

If $ik$ is a multiple root of order $r$ of $f(D) = 0$ the particular integral is the real part of

$$\frac{-icx^r e^{i(kx+\alpha)}}{f^{(r)}(ik)}.$$

**8.125** $$V(x) = ce^{kx} \cdot X,$$

where $X$ is any function of $x$.

$$y = ce^{kx} \frac{1}{f(D + k)} X.$$

If $X$ is a rational integral function of $x$ this may be evaluated by the method of **8.121.**

**8.126** $$V(x) = c \cos (kx + \alpha) \cdot X,$$

where $X$ is any function of $x$. The particular integral is the real part of

$$ce^{i(kx+\alpha)} \frac{1}{f(D + ik)} X.$$

**8.127** $$V(x) = c \sin (kx + \alpha) \cdot X.$$

The particular integral is the real part of

$$-ice^{i(kx+\alpha)} \frac{1}{f(D + ik)} X.$$

**8.128** $$V(x) = ce^{\beta x}\cos(kx + \alpha).$$

If $(\beta + ik)$ is not a root of $f(D) = 0$ the particular integral is the real part of

$$ce^{i(kx+\alpha)}\frac{1}{f(\beta + ik)}e^{\beta x}.$$

If $(\beta + ik)$ is a multiple root of order $r$ of $f(D) = 0$ the particular integral is the real part of

$$\frac{ce^{i(kx+\alpha)}x^r e^{\beta x}}{f^{(r)}(\beta + ik)},$$

where $f^{(r)}(\beta + ik)$ is formed as in **8.123.**

**8.129** $$V = ce^{\beta x}\sin(kx + \alpha).$$

If $(\beta + ik)$ is not a root of $f(D) = 0$ the particular integral is the real part of

$$\frac{-ice^{i(kx+\alpha)}e^{\beta x}}{f(\beta + ik)}.$$

If $(\beta + ik)$ is a multiple root of order $r$ of $f(D) = 0$ the particular integral is the real part of

$$\frac{-ice^{i(kx+\alpha)}x^r e^{\beta x}}{f^{(r)}(\beta + ik)}.$$

**8.130** $$V(x) = x^m X,$$

where $X$ is any function of $x$.

$$y = x^m\frac{1}{f(D)}X + mx^{m-1}\left\{\frac{d}{dD}\frac{1}{f(D)}\right\}X + \frac{m(m-1)}{2!}x^{m-2}\left\{\frac{d^2}{dD^2}\frac{1}{f(D)}\right\}X + \ldots\ldots$$

The series must be extended to the $(m + 1)$th term.

**8.200** Homogeneous linear equations. General form:

$$x^n\frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}x\frac{dy}{dx} + a_n y = V(x).$$

Denote the operator:

$$x\frac{d}{dx} = \theta,$$

$$x^m\frac{d^m}{dx^m} = \theta(\theta - 1)(\theta - 2)\ldots\ldots(\theta - m + 1).$$

The differential equation may be written:

$$F(\theta)\cdot y = V(x).$$

The complete solution is the sum of the complementary function, obtained by solving the equation with $V(x) = 0$, and the particular integral.

**8.201** The complementary function.

$$y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2} + \ldots\ldots + c_n x^{\lambda_n},$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the $n$ roots of

$$F(\lambda) = 0$$

if the roots are all distinct.

If $\lambda_k$ is a multiple root of order $r$, the corresponding terms in the complementary function are:

$$x^{\lambda_k}\{b_1 + b_2 \log x + b_3 (\log x)^2 + \ldots\ldots + b_r (\log x)^{r-1}\}.$$

If $\lambda = \mu \pm i\nu$ is a pair of complex roots, of order $r$, the corresponding terms in the complementary function are:

$$x^{\mu}\{[A_1 + A_2 \log x + A_3 (\log x)^2 + \ldots\ldots + A_r (\log x)^{r-1}] \cos (\nu \log x) + [B_1 + B_2 \log x + B_3 (\log x)^2 + \ldots\ldots + B_r (\log x)^{r-1}] \sin (\nu \log x)\}.$$

**8.202** The particular integral.

If

$$F(\theta) = (\theta - \lambda_1)(\theta - \lambda_2) \ldots\ldots (\theta - \lambda_n),$$

$$y = x^{\lambda_1} \int x^{\lambda_2 - \lambda_1 - 1} dx \int x^{\lambda_3 - \lambda_2 - 1} dx \ldots\ldots \int x^{\lambda_n - \lambda_{n-1} - 1} V(x) dx.$$

**8.203** The operator $\frac{1}{F(\theta)}$ may be resolved into partial fractions:

$$\frac{1}{F(\theta)} = \frac{N_1}{\theta - \lambda_1} + \frac{N_2}{\theta - \lambda_2} + \ldots\ldots + \frac{N_n}{\theta - \lambda_n},$$

$$y = N_1 x^{\lambda_1} \int x^{-\lambda_1 - 1} V(x) dx + N_2 x^{\lambda_2} \int x^{-\lambda_2 - 1} V(x) dx + \ldots\ldots + N_n x^{\lambda_n} \int x^{-\lambda_n - 1} V(x) dx.$$

The particular integral in special cases.

**8.210**

$$V(x) = cx^k,$$

$$y = \frac{c}{F(k)} x^k,$$

unless $k$ is a root of $F(\theta) = 0$.

If $k$ is a multiple root of order $r$ of $F(\theta) = 0$.

$$y = \frac{c (\log x)^r}{F^{(r)}(k)},$$

where $F^{(r)}(k)$ is obtained by taking the $r$th derivative of $F(\theta)$ with respect to $\theta$ and after differentiation substituting $k$ for $\theta$.

**8.211** $V(x) = cx^k X,$

where $X$ is any function of $x$. $y = cx^k \frac{1}{F(\theta + k)} X.$

**8.220** The differential equation:

$$(a + bx)^n \frac{d^n y}{dx^n} + (a + bx)^{n-1} a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + (a + bx) a_{n-1} \frac{dy}{dx} + a_n y = V(x),$$

may be reduced to the homogeneous linear equation (**8.200**) by the change of variable

$$z = a + bx.$$

It may be reduced to a linear equation with constant coefficients by the change of variable:

$$e^z = a + bx.$$

**8.230** The general linear equation. General form:

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + P_{n-1} \frac{dy}{dx} + P_n = V,$$

where $P_0, P_1, \ldots\ldots, P_n, V$ are functions of $x$ only.

The complete solution is the sum of:

(a) The complementary function, which is the general solution of the equation with $V = 0$, and containing $n$ arbitrary constants, and

(b) The particular integral.

**8.231** Complementary Function. If $y_1, y_2, \ldots\ldots, y_n$ are $n$ independent solutions of **8.230** with $V = 0$, the complementary function is

$$y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n.$$

The conditions that $y_1, y_2, \ldots\ldots, y_n$ be $n$ independent solutions is that the determinant $\Delta \neq 0$.

$$\Delta = \begin{vmatrix} \frac{d^{n-1} y_1}{dx^{n-1}} & \frac{d^{n-1} y_2}{dx^{n-1}} & \cdots\cdots & \frac{d^{n-1} y_n}{dx^{n-1}} \\ \frac{d^{n-2} y_1}{dx^{n-2}} & \frac{d^{n-2} y_2}{dx^{n-2}} & \cdots\cdots & \frac{d^{n-2} y_n}{dx^{n-2}} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots\cdots & \frac{dy_n}{dx} \\ y_1 & y_2 & \cdots\cdots & y_n \end{vmatrix}$$

When $\Delta \neq 0$:

$$\Delta = Ce^{-\int \frac{P_1}{P_0} dx}.$$

**8.232** The particular integral. If $\Delta_k$ is the minor of $\frac{d^{n-1}y_k}{dx^{n-1}}$ in $\Delta$, the particular integral is:

$$y = y_1 \int \frac{V\Delta_1}{P_0\Delta}\,dx + y_2 \int \frac{V\Delta_2}{P_0\Delta}\,dx + \ldots + y_n \int \frac{V\Delta_n}{P_0\Delta}\,dx.$$

**8.233** If $y_1$ is one integral of the equation **8.230** with $v = 0$, the substitution

$$y = uy_1, \quad v = \frac{du}{dx},$$

will result in a linear equation of order $n - 1$.

**8.234** If $y_1, y_2, \ldots, y_{n-1}$ are $n - 1$ independent integrals of **8.230** with $V = 0$ the complete solution is:

$$y = \sum_{k=1}^{n-1} y\, c_{kk} + c_n \sum_{k=1}^{n-1} y_k \int \frac{\Delta_k}{\Delta^2}\, e^{-\int \frac{P_1}{P_0}dx}\,dx$$

where $\Delta$ is the determinant:

$$\Delta = \begin{vmatrix} \frac{d^{n-2}y_1}{dx^{n-2}} & \frac{d^{n-2}y_2}{dx^{n-2}} & \cdots & \frac{d^{n-2}y_{n-1}}{dx^{n-2}} \\ \frac{d^{n-3}y_1}{dx^{n-3}} & \frac{d^{n-3}y_2}{dx^{n-3}} & \cdots & \frac{d^{n-3}y_{n-1}}{dx^{n-3}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_{n-1}}{dx} \\ y_1 & y_2 & \cdots & y_{n-1} \end{vmatrix}$$

and $\Delta_k$ is the minor of $\frac{d^{n-2}y_k}{dx^{n-2}}$ in $\Delta$.

## SYMBOLIC METHODS

**8.240** Denote the operators:

$$\frac{d}{dx} = D$$

$$x\frac{d}{dx} = \theta.$$

**8.241** If $X$ is a function of $x$:

1. $$(D - m)^{-1} X = e^{mx} \int e^{-mx} X dx.$$

2. $$(D - m)^{-1} 0 = ce^{mx}.$$

3. $$(\theta - m)^{-1} X = x^m \int x^{-m-1} X dx.$$

4. $$(\theta - m)^{-1} 0 = cx^m.$$

**8.242** If $F(D)$ is a polynomial in $D$,

1. $$F(D)e^{mx} = e^{mx}F(m).$$
2. $$F(D)e^{mx}X = e^{mx}F(D+m)X.$$
3. $$e^{mx}F(D)X = F(D-m)e^{mx}X.$$

**8.243** If $F(\theta)$ is a polynomial in $\theta$,

1. $$F(\theta)x^m = x^mF(m).$$
2. $$F(\theta)x^mX = x^mF(\theta+m)X.$$
3. $$x^mF(\theta)X = F(\theta-m)x^mX.$$

**8.244** $$x^m\frac{d^m}{dx^m} = \theta(\theta-1)\,(\theta-2)\,.\,.\,.\,.\,.\,(\theta-m+1).$$

## INTEGRATION IN SERIES

**8.250** If a linear differential equation can be expressed in the symbolic form:

$$[x^mF(\theta)+f(\theta)]\,y=0,$$

where $F(\theta)$ and $f(\theta)$ are polynomials in $\theta$, the substitution,

$$y=\sum_{n=0}^{\infty}a_nx^{\rho+nm},$$

leads to the equations,

$$\begin{aligned}
a_0f(\rho) &= 0,\\
a_0F(\rho)+a_1f(\rho+m) &= 0,\\
a_1F(\rho+m)+a_2f(\rho+2m) &= 0,\\
a_2F(\rho+2m)+a_3f(\rho+3m) &= 0.\\
&\cdots\\
&\cdots
\end{aligned}$$

**8.251** The equation

$$f(\rho)=0,$$

is the "indicial equation." If it is satisfied $a_0$ may be chosen arbitrarily, and the other coefficients are then determined.

**8.252** An equation:

$$\left[F(\theta)+\phi(\theta)\frac{d^m}{dx^m}\right]y=0,$$

may be reduced to the form **8.250**, where,

$$f(\theta)=\phi(\theta-m)\,\theta(\theta-1)\,(\theta-2)\,.\,.\,.\,.\,(\theta-m+1).$$

If the degree of the polynomial $f$ is greater than that of $F$ the series always converges; if the degree of $f$ is less than that of $F$ the series always diverges.

## ORDINARY DIFFERENTIAL EQUATIONS OF SPECIAL TYPES

**8.300**

$$\frac{d^n y}{dx^n} = X,$$

where $X$ is a function of $x$ only.

$$y = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} T dt + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1} x + c_n,$$

where $T$ is the same function of $t$ that $X$ is of $x$.

**8.301**

$$\frac{d^2 y}{dx^2} = Y,$$

where $Y$ is a function of $y$ only.

If

$$\psi(y) = 2\int Y dy,$$

the solution is:

$$\int \frac{dy}{\{\psi(y) + c_1\}^{\frac{1}{2}}} = x + c_2.$$

**8.302**

$$\frac{d^n y}{dx^n} = F\left(\frac{d^{n-1} y}{dx^{n-1}}\right).$$

Put

$$\frac{d^{n-1} y}{dx^{n-1}} = Y; \quad \frac{dY}{dx} = F(Y),$$

$$x + c_1 = \int \frac{dY}{F(Y)} = \psi(Y),$$

$$Y = \phi(x + c_1),$$

$$\frac{d^{n-1} y}{dx^{n-1}} = \phi(x + c_1),$$

and this equation may be solved by **8.300.**

Or the equation can be solved:

$$y = \int \frac{dY}{F(Y)} \int \frac{dY}{F(Y)} \cdot \cdots \cdots \int \frac{Y dY}{F(Y)},$$

where the integration is to be carried out from right to left and an arbitrary constant added after each integration. Eliminating $Y$ between this result and

$$Y = \phi(x + c_1)$$

gives the solution.

**8.303**

$$\frac{d^n y}{dx^n} = F\left(\frac{d^{n-2} y}{dx^{n-2}}\right).$$

Put

$$\frac{d^{n-2}y}{dx^{n-2}} = Y,$$

$$\frac{d^2Y}{dx^2} = F(Y),$$

which may be solved by **8.301.** If the solution can be expressed:

$$Y = \phi(x),$$

$n - 2$ integrations will solve the given differential equation.

Or putting

$$\psi(y) = 2\int Y\,dy,$$

$$y = \int \frac{dY}{\{c_1 + \psi(Y)\}^{\frac{1}{2}}} \int \frac{dY}{\{c_1 + \psi(Y)\}^{\frac{1}{2}}} \cdots\cdots \int \frac{Y\,dY}{\{c_1 + \psi(Y)\}^{\frac{1}{2}}},$$

where the integration is to be carried out from right to left and an arbitrary constant added after each integration. The solution of the given differential equation is obtained by elimination between this result and

$$Y = \phi(x).$$

**8.304** Differential equations of the second order in which the independent variable does not appear. General type:

$$F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

Put

$$p = \frac{dy}{dx}, \quad p\frac{dp}{dy} = \frac{d^2y}{dx^2}.$$

A differential equation of the first order results:

$$F\left(y, p, p\frac{dp}{dy}\right) = 0.$$

If the solution of this equation is:

$$p = f(y),$$

the solution of the given equation is,

$$x + c_2 = \int \frac{dy}{f(y)}.$$

**8.305** Differential equations of the second order in which the dependent variable does not appear. General type:

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

Put

$$p = \frac{dy}{dx}, \quad \frac{dp}{dx} = \frac{d^2y}{dx^2}.$$

A differential equation of the first order results:

$$F\left(x, p, \frac{dp}{dx}\right) = 0.$$

If the solution of this equation is:

$$p = f(x),$$

the solution of the given equation is:

$$y = c_2 + \int f(x)dx.$$

**8.306** Equations of an order higher than the second in which either the independent or the dependent variable does not appear. The substitution:

$$\frac{dy}{dx} = p,$$

as in **8.304** and **8.305** will result in an equation of an order less by unity than the given equation.

**8.307** Homogeneous differential equations. If $y$ is assumed to be of dimensions $n$, $x$ of dimensions 1, $\frac{dy}{dx}$ of dimensions $(n-1)$, $\frac{d^2y}{dx^2}$ of dimensions $(n-2)$, . . . . . then if every term has the same dimensions the equation is homogeneous. If the independent variable is changed to $\theta$ and the dependent variable changed to $z$ by the relations,

$$x = e^{\theta}, \quad y = ze^{n\theta},$$

the resulting equation will be one in which the independent variable does not appear and its order can be lowered by unity by **8.306.**

If $y$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, . . . . are assumed all to be of the same dimensions, and the equation is homogeneous, the substitution:

$$y = e^{\int u dx},$$

will result in an equation in $u$ and $x$ of an order less by unity than the given equation.

**8.310** Exact differential equations. A linear differential equation:

$$P_n \frac{d^n y}{dx^n} + P_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_1 \frac{dy}{dx} + P_0 = P,$$

where $P, P_0, P_1, \ldots . P_n$ are functions of $x$ is exact if:

$$P_0 - \frac{dP_1}{dx} + \frac{d^2P_2}{dx^2} - \ldots + (-1)^n \frac{d^nP_n}{dx^n} = 0.$$

The first integral is:

$$Q_n \frac{d^{n-1}}{dx^{n-1}} + Q_{n-1} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + Q_1 y = \int P\,dx + c_1,$$

where,

$$Q_n = P_n,$$

$$Q_{n-1} = P_{n-1} - \frac{dP_n}{dx},$$

$$Q_{n-2} = P_{n-2} - \frac{dP_{n-1}}{dx} + \frac{d^2P_n}{dx^2},$$

. . . .

. . . .

$$Q_1 = P_1 - \frac{dP_2}{dx} + \frac{d^2P_3}{dx^2} - \ldots + (-1)^{n-1} \frac{d^{n-1}P_n}{dx^{n-1}}.$$

If the first integral is an exact differential equation the process may be continued as long as the coefficients of each successive integral satisfy the condition of integrability.

**8.311** Non-linear differential equations. A non-linear differential equation of the $n$th order:

$$V\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y, x\right) = 0,$$

to be exact must contain $\frac{d^ny}{dx^n}$ in the first degree only. Put

$$\frac{d^{n-1}y}{dx^{n-1}} = p, \quad \frac{d^ny}{dx^n} = \frac{dp}{dx}.$$

Integrate the equation on the assumption that $p$ is the only variable and $\frac{dp}{dx}$ its differential coefficient. Let the result be $V_1$. In $V\,dx - dV_1$, $\frac{d^{n-1}y}{dx^{n-1}}$ is the highest differential coefficient and it occurs in the first degree only. Repeat this process as often as may be necessary and the first integral of the exact differential equation will be

$$V_1 + V_2 + \ldots = c.$$

If this process breaks down owing to the appearance of the highest differential coefficient in a higher degree than the first the given differential equation was not exact.

**8.312** General condition for an exact differential equation. Write:

$$\frac{dy}{dx} = y' \quad \frac{d^2y}{dx^2} = y'' \ldots\ldots \quad \frac{d^ny}{dx^n} = y^{(n)}.$$

In order that the differential equation:

$$V(x, y, y', y'', \ldots\ldots, y^{(n)}) = 0,$$

be exact it is necessary and sufficient that

$$\frac{\partial V}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial V}{\partial y''}\right) - \ldots\ldots + (-1)^n \frac{\partial^n}{\partial x^n}\left(\frac{\partial V}{\partial y^{(n)}}\right) = 0.$$

**8.400** Linear differential equations of the second order.

General form:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

where $P$, $Q$, $R$ are, in general, functions of $x$.

**8.401** If a solution of the equation with $R = 0$:

$$y = w$$

can be found, the complete solution of the given differential equation is:

$$y = c_2 w + c_1 w \int e^{-\int P dx} \frac{dx}{w^2} + w \int e^{-\int P dx} \frac{dx}{w^2} \int wRe^{\int P dx}\, dx.$$

**8.402** The general linear differential equation of the second order may be reduced to the form:

$$\frac{d^2v}{dx^2} + Iv = Re^{\frac{1}{2}\int P dx},$$

where:

$$y = ve^{-\frac{1}{2}\int P dx},$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2.$$

**8.403** The differential equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

by the change of independent variable to

$$z = \int e^{-\int P dx}\, dx,$$

becomes:

$$\frac{d^2y}{dz^2} + Qe^{2\int P dx} y = 0.$$

By the change of independent variable.

$$dz = Qe^{\int P dx}\, dx,$$

$$Qe^{2\int P dx} = \frac{1}{U(z)},$$

it becomes:

$$\frac{d}{dz}\left\{\frac{1}{U}\frac{dy}{dz}\right\} + y = 0.$$

**8.404** Resolution of the operator. The differential equation:

$$u\frac{d^2y}{dx^2} + v\frac{dy}{dx} + wy = 0,$$

may sometimes be solved by resolving the operator,

$$u\frac{d^2}{dx^2} + v\frac{d}{dx} + w,$$

into the product,

$$\left(p\frac{d}{dx} + q\right)\left(r\frac{d}{dx} + s\right).$$

The solution of the differential equation reduces to the solution of

$$r\frac{dy}{dx} + sy = c_1 e^{-\int \frac{q}{p}dx}.$$

The equations for determining $p$, $r$, $q$, $s$ are:

$$pr = u,$$

$$qr + ps + p\frac{dr}{dx} = v,$$

$$qs + p\frac{ds}{dx} = w.$$

**8.410** Variation of parameters. The complete solution of the differential equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

is

$$y = c_1 f_2(x) + c_2 f_1(x) + \frac{1}{C}\int^x R(\xi) e^{\int^\xi P dx}\left\{ f_2(x)f_1(\xi) - f_1(x)f_2(\xi) \right\} d\xi,$$

where $f_1(x)$ and $f_2(x)$ are two particular solutions of the differential equation with $R = 0$, and are therefore connected by the relation

$$f_1\frac{df_2}{dx} - f_2\frac{df_1}{dx} = Ce^{-\int P dx}.$$

$C$ is an absolute constant depending upon the forms of $f_1$ and $f_2$ and may be taken as unity.

**8.500** The differential equation:

$$(a_2 + b_2x)\frac{d^2y}{dx^2} + (a_1 + b_1x)\frac{dy}{dx} + (a_0 + b_0x)y = 0.$$

**8.501** Let

$$D = (a_0b_1 - a_1b_0)(a_1b_2 - a_2b_1) - (a_0b_2 - a_2b_0)^2.$$

Special cases.

**8.502** $b_2 = b_1 = b_0 = 0$.

The solution is:

$$y_1 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},$$

where:

$$\frac{\lambda_1}{\lambda_2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}.$$

**8.503** $D = 0$, $b_2 = 0$,

$$y = e^{\lambda x}\left\{ c_1 + c_2 \int e^{-(k+2\lambda)x - mx^2} dx \right\},$$

where:

$$k = \frac{a_1}{a_2} \quad m = \frac{b_1}{2a_2} \quad \lambda = -\frac{b_0}{b_1}.$$

**8.504** $D = 0$, $b_2 \neq 0$:

$$y = e^{\lambda x}\left\{ c_1 + c_2 \int e^{-(k+2\lambda)x}(a_2 + b_2 x)^m dx \right\},$$

where

$$k = \frac{b_1}{b_2} \quad m = \frac{a_2b_1 - a_1b_2}{b_2^3},$$

and $\lambda$ is the common root of:

$$a_2\lambda^2 + a_1\lambda + a_0 = 0,$$
$$b_2\lambda^2 + b_1\lambda + b_0 = 0.$$

**8.505** $D \neq 0$, $b_2 = b_1 = 0$. If $\eta = f(\xi)$ is the complete solution of:

$$\frac{d^2\eta}{d\xi^2} + \xi\eta = 0,$$

$$y = e^{\lambda x} f\left(\frac{\alpha + \beta x}{\beta^{\frac{2}{3}}}\right),$$

where

$$\alpha = \frac{4a_0a_2 - a_1^2}{4a_2^2} \quad \beta = \frac{b_0}{a_2} \quad \lambda = -\frac{a_1}{2a_2}.$$

**8.510** The differential equation **8.500** under the condition $D \neq 0$ can always be reduced to the form:

$$\xi \frac{d^2\phi}{d\xi^2} + (p + q + \xi)\frac{d\phi}{d\xi} + p\phi = 0.$$

**8.511** Denote the complete solution of **8.510**:

$$\phi = F\{\xi\}.$$

**8.512** $b_2 = b_1 = 0$:

$$y = e^{\lambda x + (\mu + \nu x)^{\frac{3}{2}}} F\{2(\mu + \nu x)^{\frac{3}{2}}\},$$

where:

$$\lambda = -\frac{a_1}{2a_2} \quad \mu = \frac{a_1^2 - 4a_0a_2}{4a_2^2}\left(\frac{4a_2^2}{9b_0^2}\right)^{\frac{1}{3}},$$

$$\nu = -\left(\frac{4b_0}{9a_2}\right)^{\frac{1}{3}},$$

$$p = q = \frac{1}{6}.$$

**8.513** $b_2 = 0$, $b_1 \neq 0$:

$$y = e^{\lambda x} F\left\{ \frac{(\alpha_1 + \beta_1 x)^2}{2\beta_1} \right\},$$

where:

$$\lambda = -\frac{b_0}{b_1} \quad \alpha_1 = \frac{a_1 b_1 - 2a_2 b_0}{a_2 b_1}, \quad \beta_1 = \frac{b_1}{a_2},$$

$$p = \frac{a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2}{2b_1^3},$$

$$q = \frac{1}{2} - p.$$

**8.514** $b_2 \neq 0$, $b_0 = \frac{b_1^2}{4b_2}$:

$$y = e^{\lambda x + \sqrt{\mu + \nu x}} F\left\{ 2\sqrt{\mu + \nu x} \right\},$$

where:

$$\lambda = -\frac{b_1}{2b_2}, \quad \mu = -a_2 \frac{4a_0 b_2^2 - 2a_1 b_1 b_2 + a_2 b_1^2}{b_2^4},$$

$$\nu = -\frac{4a_0 b_2^2 - 2a_1 b_1 b_2 + a_2 b_1^2}{b_2^3},$$

$$p = q = \frac{a_1 b_2 - a_2 b_1}{b_2^2} - \frac{1}{2}.$$

**8.515** $b_2 \neq 0$, $b_0 \neq \frac{b_1^2}{4b_2}$:

$$y = e^{\lambda x} F\left\{ \frac{\beta_1(\alpha_2 + \beta_2 x)}{\beta_2^2} \right\},$$

where $\alpha_2 = a_2$, $\beta_2 = b_2$, $\beta_1 = 2b_2\lambda + b_1$ and $\lambda$ is one of the roots of

$$b_2\lambda^2 + b_1\lambda + b_0 = 0.$$

$$p = \frac{a_2\lambda^2 + a_1\lambda + a_0}{2b_2\lambda + b_1}, \qquad q = \frac{a_1 b_2 - a_2 b_1}{b_2^2} - p.$$

**8.520** The solution of **8.510** will be denoted:

$$\phi = F(p, q, \xi).$$

1. $F(p, q, \xi) = e^{-\xi} F(q, p, -\xi).$
2. $F(p, q, -\xi) = e^{\xi} F(q, p, \xi)$
3. $F(q, p, \xi) = e^{-\xi} F(p, q, -\xi).$
4. $F(p, q, \xi) = \xi^{1-p-q} F(1 - q, 1 - p, \xi).$
5. $F(-p, -q, \xi) = \xi^{1+p+q} F(1 + q, 1 + p, \xi).$
6. $F(p + m, q, \xi) = \frac{d^m}{d\xi^m} F(p, q, \xi).$
7. $F(p, q + n, \xi) = (-1)^n e^{-\xi} \frac{d^n}{d\xi^n} \left\{ e^{\xi} F(p, q, \xi) \right\}.$

**8.521** The function $F(p, q, \xi)$ can always be found if it is known for positive proper fractional values of $p$ and $q$.

**8.522** $p$ and $q$ positive improper fractions:

$$p = m + r, \quad q = n + s$$

where $m$ and $n$ are positive integers and $r$ and $s$ positive proper fractions.

$$F(m + r, n + s, \xi) = (-1)^n \frac{d^m}{d\xi^m}\left[e^{-\xi}\frac{d^n}{d\xi^n}\left\{e^{\xi} F(r, s, \xi)\right\}\right].$$

**8.523** $p$ and $q$ both negative:

$$p = -(m - 1 + r) \quad q = -(n - 1 + s),$$

$$F(-m + 1 - r, -n + 1 - s, \xi) = (-1)^m \xi^{m+n+r+s-1} \frac{d^n}{d\xi^n}\left[e^{-\xi}\frac{d^m}{d\xi^m}\left\{e^{\xi} F(s, r, \xi)\right\}\right].$$

**8.524** $p$ positive, $q$ negative:

$$p = m + r, \quad q = -n + s,$$

$$F(m + r, -n + s, \xi) = \frac{d^m}{d\xi^m}\left[\xi^{n+1-r-s}\frac{d^n}{d\xi^n} F(1 - s, 1 - r, \xi)\right].$$

**8.525** $p$ negative, $q$ positive:

$$p = -m + r, \quad q = n + s,$$

$$F(-m + r, n + s, \xi) = (-1)^{m+n} e^{-\xi}\frac{d^n}{d\xi^n}\left[\xi^{m+1-r-s}\frac{d^m}{d\xi^m}\left\{e^{\xi} F(1 - s, 1 - r, \xi)\right\}\right].$$

**8.530** If either $p$ or $q$ is zero the relation $D = 0$ is satisfied and the complete solution of the differential equation is given in **8.502, 3.**

**8.531** If $p = m$, a positive integer:

$$\phi = F(m, q, \xi) = c_1 \frac{d^{m-1}}{d\xi^{m-1}}\left[\xi^{-q} e^{-\xi}\int \xi^{q-1} e^{\xi} d\xi\right] + c_2 \frac{d^{m-1}}{d\xi^{m-1}}\left[\xi^{-q} e^{-\xi}\right].$$

**8.532** If $p = m$, a positive integer and both $q$ and $\xi$ are positive:

$$\phi = F(m, q, \xi) = c_1 \int_0^1 u^{m-1}(1 - u)^{q-1} e^{-\xi u}\, du + c_2 e^{-\xi}\int_0^{\infty} (1 + u)^{m-1} u^{q-1} e^{-\xi u}\, du.$$

**8.533** If $q = n$, a positive integer:

$$\phi = F(p, n, \xi) = c_1 e^{-\xi}\frac{d^{n-1}}{d\xi^{n-1}}\left[\xi^{-p} e^{\xi}\int \xi^{p-1} e^{-\xi}\, d\xi\right] + c_2 e^{-\xi}\frac{d^{n-1}}{d\xi^{n-1}}\left[\xi^{-p} e^{\xi}\right].$$

**8.534** If $q = n$, a positive integer and both $p$ and $\xi$ are positive:

$$\phi = F(p, n, \xi) = c_1 \int_0^1 u^{p-1}(1 - u)^{n-1} e^{-\xi u}\, du + c_2 e^{-\xi}\int_0^{\infty} (1 + u)^{p-1} u^{n-1} e^{-\xi u}\, du.$$

**8.540** The general solution of equation **8.510** may be written:

$$\phi = F(p, q, \xi) = c_1 M + c_2 N,$$

$$M = \int_0^1 u^{p-1}(1-u)^{q-1}e^{-\xi u}du \qquad \begin{matrix} p>0 \\ q>0 \end{matrix}$$

$$N = \int_0^\infty (1+u)^{p-1}u^{q-1}e^{-\xi(1+u)}du \qquad \begin{matrix} q>0 \\ \xi>0 \end{matrix}$$

$$M = \frac{\Gamma(p)\Gamma(q)}{\Gamma(s)}\left\{1 - \frac{p}{s}\frac{\xi}{1!} + \frac{p(p+1)}{s(s+1)}\frac{\xi^2}{2!} - \frac{p(p+1)(p+2)}{s(s+1)(s+2)}\frac{\xi^3}{3!} + \cdots\right\}$$

$$s = p + q,$$

$$N = \frac{\Gamma(q)e^{-\xi}}{\xi^q}\left\{1 + \frac{(p-1)q}{1!\xi} + \frac{(p-1)(p-2)q(q+1)}{2!\xi^2} + \cdots\right.$$

$$+ \frac{(p-1)(p-2)\cdots(p-n-1)(q)(q+1)\cdots(q+n-2)}{(n-1)!\xi^{n-1}}$$

$$\left. + \frac{\rho(p-1)(p-2)\cdots(p-n)q(q+1)(q+2)\cdots(q+n-1)}{n!\xi^n}\right\},$$

where $0 < \rho < 1$ and the real part of $\xi$ is positive.

## THE COMPLETE SOLUTION OF EQUATION **8.510** IN SPECIAL CASES

**8.550** $p>0$, $q>0$, real part of $\xi>0$:

$$F(p, q, \xi) = c_1\int_0^1 u^{p-1}(1-u)^{q-1}e^{-\xi u}du + c_2e^{-\xi}\int_0^\infty (1+u)^{p-1}u^{q-1}e^{-\xi u}du.$$

**8.551** $p>0$, $q>0$, $\xi<0$:

$$F(p, q, \xi) = c_1\int_0^1 u^{p-1}(1-u)^{q-1}e^{-\xi u}du + c_2\int^\infty u^{p-1}(1+u)^{q-1}e^{\xi u}du.$$

**8.552** $p<0$, $q<0$, $\xi>0$:

$$F(p, q, \xi) = \xi^{1-p-q}\left\{c_1\int_0^1 (1-u)^{-p}u^{-q}e^{-\xi u}du + c_2e^{-\xi}\int_0^\infty u^{-p}(1+u)^{-q}e^{-\xi u}du\right\}.$$

**8.553** $p<0$, $q<0$, $\xi<0$:

$$F(p, q, \xi) = \xi^{1-p-q}\left\{c_1\int_0^1 (1-u)^{-p}u^{-q}e^{-\xi u}du + c_2\int_0^\infty (1+u)^{-p}u^{-q}e^{+\xi u}du\right\}.$$

**8.554** $p>0$, $q<0$

$p = m + r$, where $m$ is a positive integer and $r$ a proper fraction.

$$F(m+r, q, \xi) = \frac{d^m}{d\xi^m}\left\{\xi^{1-r-q}F(1-r, 1-q, \xi)\right\},$$

$\xi>0$: $F(1-r, 1-q, \xi) = c_1 \int_0^1 u^{-r}(1-u)^{-q}e^{-\xi u}\,du$

$$+ c_2 e^{-\xi} \int_0^\infty (1+u)^{-r}u^{-q}e^{-\xi u}\,du,$$

$\xi<0$: $F(1-r, 1-q, \xi) = c_1 \int_0^1 u^{-r}(1-u)^{-q}e^{-\xi u}\,du$

$$+ c_2 \int_0^\infty u^{-r}(1+u)^{-q}e^{\xi u}\,du.$$

**8.555** $p<0$, $q>0$,

$q = n + s$, where $n$ is a positive integer and $s$ a proper fraction.

$$F(p, n+s, \xi) = e^{-\xi}\frac{d^n}{d\xi^n}\left\{ e^{\xi}\xi^{1-p-s}F(1-s, 1-p, \xi)\right\},$$

$\xi>0$: $F(1-s, 1-p, \xi) = c_1 \int_0^1 u^{-s}(1-u)^{-p}e^{-\xi u}\,du$

$$+ c_2 e^{-\xi} \int_0^\infty (1+u)^{-s}u^{-p}e^{-\xi u}\,du,$$

$\xi<0$: $F(1-s, 1-p, \xi) = c_1 \int_0^1 u^{-s}(1-u)^{-p}e^{-\xi}\,du$

$$+ c_2 \int_0^\infty u^{-s}(1+u)^{-p}e^{\xi u}du.$$

**8.556** $\xi$ pure imaginary:

$p = r$, $q = s$, where $r$ and $s$ are positive proper fractions.

$r + s \neq 1$:

$$F(r, s, \xi) = c_1 \int_0^1 u^{r-1}(1-u)^{s-1}e^{-\xi u}\,du$$

$$+ c_2\xi^{1-r-s} \int_0^1 u^{-s}(1-u)^{-r}e^{-\xi u}\,du.$$

$r + s = 1$:

$$F(r, s, \xi) = c_1 \int_0^1 u^{r-1}(1-u)^{s-1}e^{-\xi u}\,du$$

$$+ c_2 \int_0^1 u^{r-1}(1-u)^{s-1}e^{-\xi u}\log\left\{ \xi u(1-u)\right\} du.$$

**8.600** The differential equation:

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0$$

is satisfied by the confluent hypergeometric function. The complete solution is:

$$y = c_1 M(\alpha, \gamma, x) + c_2 x^{1-\gamma} M(\alpha - \gamma + 1, 2 - \gamma, x) = \overline{M}(\alpha, \gamma, x),$$

where

$$M(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \cdots\cdots$$

The series is absolutely and uniformly convergent for all real and complex values of $\alpha$, $\gamma$, $x$, except when $\gamma$ is a negative integer or zero.

When $\gamma$ is a positive integer the complete solution of the differential equation is:

$$y = \left\{ c_1 + c_2 \log x \right\} M(\alpha, \gamma, x) + c_2 \left\{ \frac{ax}{\gamma}\left(\frac{1}{\alpha} - \frac{1}{\gamma} - 1\right) \right.$$

$$+ \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - 1 - \frac{1}{2}\right)$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\alpha+2} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - \frac{1}{\gamma+2} - 1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$\left. + \cdots\cdots \right\}.$$

**8.601** For large values of $x$ the following asymptotic expansion may be used: $M(\alpha, \gamma, x)$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-x)^{-\alpha}\left\{ 1 - \frac{\alpha(\alpha-\gamma+1)}{1}\frac{1}{x} + \frac{\alpha(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma+2)}{2!}\frac{1}{x^2} \cdots \right\}$$

$$+ \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^x x^{\alpha-\gamma}\left\{ 1 + \frac{(1-\alpha)(\gamma-\alpha)}{1}\frac{1}{x} + \frac{(1-\alpha)(2-\alpha)(\gamma-\alpha)(\gamma-\alpha+1)}{2!}\frac{1}{x^2} + \cdots \right\}.$$

**8.61**

1. $M(\alpha, \gamma, x) = e^x M(\gamma-\alpha, \gamma, -x)$.
2. $x^{1-\gamma} M(\alpha-\gamma+1, 2-\gamma, x) = e^x x^{1-\gamma} M(1-\alpha, 2-\gamma, -x)$.
3. $\frac{x}{\gamma} M(\alpha+1, \gamma+1, x) = M(\alpha+1, \gamma, x) - M(\alpha, \gamma, x)$.
4. $\alpha M(\alpha+1, \gamma+1, x) = (\alpha-\gamma)M(\alpha, \gamma+1, x) + \gamma M(\alpha, \gamma, x)$.
5. $(\alpha+x)M(\alpha+1, \gamma+1, x) = (\alpha-\gamma)M(\alpha, \gamma+1, x) + \gamma M(\alpha+1, \gamma, x)$.
6. $\alpha\gamma M(\alpha+1, \gamma, x) = \gamma(\alpha+x)M(\alpha, \gamma, x) - x(\gamma-\alpha)M(\alpha, \gamma+1, x)$.
7. $\alpha M(\alpha+1, \gamma, x) = (x+2\alpha-\gamma)M(\alpha, \gamma, x) + (\gamma-\alpha)M(\alpha-1, \gamma, x)$.
8. $\frac{\gamma-\alpha}{\gamma} xM(\alpha, \gamma+1, x) = (x+\gamma-1)M(\alpha, \gamma, x) + (1-\gamma)M(\alpha, \gamma-1, x)$.

**8.62**

1. $\frac{d}{dx} M(\alpha, \gamma, x) = \frac{\alpha}{\gamma} M(\alpha+1, \gamma+1, x)$.
2. $(1-\alpha)\int_0^x M(\alpha, \gamma, x)\,dx = (1-\gamma)M(\alpha-1, \gamma-1, x) + (\gamma-1)$.

SPECIAL DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS IN TERMS OF $\overline{M}(\alpha, \gamma, x)$

**8.630**

$$\frac{d^2y}{dx^2} + 2(p+qx)\frac{dy}{dx} + \left\{ 4\alpha q + p^2 - q^2m^2 + 2qx(p+qm) \right\} y = 0,$$

$$y = e^{-(p+qm)x}\,\overline{M}\left(\alpha, \tfrac{1}{2}, -q(x-m)^2\right).$$

**8.631**

$$\frac{d^2y}{dx^2} + \left(2p + \frac{\gamma}{x}\right)\frac{dy}{dx} + \left\{ p^2 - t^2 + \frac{1}{x}\left(\gamma p + \gamma t - 2\alpha t\right) \right\} y = 0,$$

$$y = e^{-(p+t)x}\,\overline{M}(\alpha, \gamma, 2tx).$$

**8.632**

$$\frac{d^2y}{dx^2} + 2(p+qx)\frac{dy}{dx} + \left\{ q + c(1-4\alpha) + (p+qx)^2 - c^2(x-m)^2 \right\} y = 0,$$

$$y = e^{-px - \frac{1}{2}qx^2 - \frac{1}{2}c(x-m)^2}\,\overline{M}\left(\alpha, \tfrac{1}{2}, c(x-m)^2\right).$$

**8.633**

$$\frac{d^2y}{dx^2} + \left(2p + \frac{q}{x}\right)\frac{dy}{dx} + \left\{ p^2 - t^2 + \frac{1}{x}\left(pq + \gamma t - 2\alpha t\right) + \frac{1}{4x^2}\left(\gamma - q\right)\left(2 - q - \gamma\right) \right\} y = 0,$$

$$y = e^{-(p+t)x}\,x^{\frac{\gamma-q}{2}}\,\overline{M}\,(\alpha, \gamma, 2tx).$$

**8.634**

$$\frac{d^2y}{dx^2} + \left\{ \frac{2\gamma - 1}{x} + 2\alpha + 2(b-c)x \right\}\frac{dy}{dx}$$

$$+ \left\{ \frac{\alpha(2\gamma - 1)}{x} + (a^2 + 2b\gamma - 4\alpha c) + 2a(b-c)x + b(b-2c)x^2 \right\} y = 0,$$

$$y = e^{-ax - \frac{1}{2}bx^2}\,\overline{M}(\alpha, \gamma, cx^2).$$

**8.635**

$$\frac{d^2y}{dx^2} + \frac{1}{x}\left(2px^r + qr - r + 1\right)\frac{dy}{dx}$$

$$+ \frac{1}{x^2}\left\{ (p^2 - t^2)x^{2r} + r(pq + \gamma t - 2\alpha t)x^r + \frac{1}{4}r^2(\gamma - q)(2 - q - \gamma) \right\} y = 0,$$

$$y = e^{-\frac{(p+t)}{r}x^r}\,x^{\frac{r}{2}(\gamma-q)}\,\overline{M}\left(\alpha, \gamma, \frac{2tx^r}{r}\right).$$

**8.640** Tables and graphs of the function $M(\alpha, \gamma, x)$ are given by Webb and Airey (Phil. Mag. 36, p. 129, 1918) for getting approximate numerical solu-

tions of any of these differential equations. The range in $x$ is 1 to 10; in $\alpha$, +0.5 to +4.0 and −0.5 to −3.0; in $\gamma$, 1 to 7. For negative values of $x$ the equations of **8.61** may be used.

## SPECIAL DIFFERENTIAL EQUATIONS

**8.700**

$$\frac{d^2y}{dx^2} + n^2y = X(x)$$

where $X(x)$ is any function of $x$. The complete solution is:

$$y = c_1e^{nx} + c_2e^{-nx} + \frac{1}{n}\int^x X(\xi)\sinh n(x-\xi)\,d\xi.$$

**8.701**

$$\frac{d^2y}{dx^2} + \kappa\frac{dy}{dx} + n^2y = X(x).$$

The complete solution, satisfying the conditions:

$$x = 0 \qquad y = y_0,$$

$$x = 0 \qquad \frac{dy}{dx} = y_0',$$

$$y = e^{-\frac{1}{2}\kappa x}\left\{ y_0'\frac{\sin n'x}{n'} + y_0\left(\cos n'x + \frac{\kappa}{2n'}\sin n'x\right)\right\}$$

$$+\frac{1}{n'}\int_0^x e^{-\frac{1}{2}\kappa(x-\xi)}\sin n'(x-\xi)X(\xi)\,d\xi,$$

where $$n' = \sqrt{n^2 - \frac{\kappa^2}{4}}.$$

**8.702**

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)\left(\frac{dy}{dx}\right)^2 = 0,$$

$$y = \int \frac{e^{-\int f(x)dx}\,dx}{\int e^{-\int f(x)dx}\,g(x)\,dx + c_1} + c_2.$$

**8.703**

$$\frac{d^2y}{dx^2} + f(y)\left(\frac{dy}{dx}\right)^2 + g(y) = 0,$$

$$x = \pm\int \frac{e^{\int f(y)dy}\,dy}{\{c_1 - 2\int e^{2\int f(y)dy}\,g(y)\,dy\}^{\frac{1}{2}}} + c_2.$$

**8.704**

$$\frac{d^2y}{dx^2} + f(y)\frac{dy}{dx} + g(y)\left(\frac{dy}{dx}\right)^2 = 0,$$

$$x = \int \frac{e^{\int g(y)dy}\,dy}{c_1 - \int e^{\int g(y)dy}f(y)\,dy} + c_2.$$

**8.705**

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(y)\left(\frac{dy}{dx}\right)^2 = 0,$$

$$\int e^{\int g(y)dy}\,dy = c_1 \int e^{-\int f(x)dx}\,dx + c_2.$$

**8.706**

$$\frac{d^2y}{dx^2} + (a + bx)\frac{dy}{dx} + abxy = 0.$$

$$y = e^{-ax}\{c_1 + c_2 \int e^{ax - \frac{1}{2}bx^2}\,dx\}.$$

**8.707**

$$x\frac{d^2y}{dx^2} + (a + bx)\frac{dy}{dx} + aby = 0,$$

$$y = e^{-bx}\{c_1 + c \int x^{-a}e^{bx}\,dx\}.$$

**8.708**

$$\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \frac{b}{x^2}\,y = 0.$$

1. $(a-1)^2 > 4b$; $\quad \lambda = \frac{1}{2}\sqrt{(a-1)^2 - 4b}$.

$$y = x^{-\frac{a-1}{2}}\{c_1 x + c_2 x^{-\lambda}\}.$$

2. $(a-1)^2 < 4b$; $\quad \lambda = \frac{1}{2}\sqrt{4b - (a-1)^2}$

$$y = x^{-\frac{a-1}{2}}\{c_1 \cos(\lambda \log x) + c_2 \sin(\lambda \log x)\}.$$

3. $(a-1)^2 = 4b$

$$y = x^{-\frac{a-1}{2}}(c_1 + c_2 \log x).$$

**8.709**

$$\frac{d^2y}{dx^2} + 2bx\frac{dy}{dx} + (a + b^2x^2\,y = 0.$$

1. $a < b$, $\quad \lambda = \sqrt{b - a}$,

$$y = e^{-\frac{bx^2}{2}}(c_1 e^{\lambda x} + c_2 e^{-\lambda x}).$$

2. $a > b$, $\quad \lambda = \sqrt{a - b}$,

$$y = e^{-\frac{bx^2}{2}}(c_1 \cos \lambda x + c_2 \sin \lambda x).$$

**8.710**

$$f(x)\frac{d^2y}{dx^2} - (a + bx)\frac{dy}{dx} + by = 0,$$

$$\int \frac{a + bx}{f(x)}\,dx = X,$$

$$y = c_1(a + bx) + c_2\left\{e^X - (a + bx)\int \frac{1}{f(x)}e^X\,dx\right\}.$$

**8.711**

$$(a^2 - x^2)\frac{d^2y}{dx^2} + 2(\mu - 1)x\frac{dy}{dx} - \mu(\mu - 1)y = 0,$$

$$y = (a + x)_\mu \left\{ c_1 + c_2 \int \frac{(a - x)^{\mu-1}}{(a + x)^{\mu+1}}\, dx \right\}.$$

**8.712**

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \mu^2 y = \frac{a}{x},$$

$$y = \frac{1}{x}\left\{ c_1 \cos \mu x + c_2 \sin \mu x + \frac{a}{\mu^2} \right\}.$$

**8.713**

$$\frac{d^4y}{dx^4} + 2d\frac{d^3y}{dx^3} + c\frac{d^2y}{dx^2} + 2b\frac{dy}{dx} + ay = 0,$$

$$y = c_1 e^{-\rho_1 x}\{\rho_1 \sin(\omega_1 x + \alpha_1) + \omega_1 \cos(\omega_1 x + \alpha_1)\}$$
$$+ c_2 e^{-\rho_2 x}\{\rho_2 \sin(\omega_2 x + \alpha_2) + \omega_2 \cos(\omega_2 x + \alpha_2)\},$$

where:

$$4\omega_1^2 = z + c - 2d^2 + 2\sqrt{z^2 - 4a} - 2d\sqrt{z - c + d^2},$$
$$4\omega_2^2 = z + c - 2d^2 - 2\sqrt{z^2 - 4a} + 2d\sqrt{z - c + d^2},$$
$$2\rho_1 = d + \sqrt{z - c + d^2},$$
$$2\rho_2 = d - \sqrt{z - c + d^2},$$

and $z$ is a root of

$$z^3 - cz^2 - 4(a - bd)z + 4(ac - ad^2 - b^2) = 0.$$

(Kiebitz, Ann. d. Physik, 40, p. 138, 1913)

# IX. DIFFERENTIAL EQUATIONS
## (*continued*)

**9.00** Legendre's Equation:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

**9.001** If $n$ is a positive integer one solution is the Legendre polynomial, or Zonal Harmonic, $P_n(x)$:

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left\{ x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \ldots . \right\}.$$

**9.002** If $n$ is even the last term in the finite series in the brackets is:

$$(-1)^{\frac{n}{2}}\frac{(n!)^3}{\left(\frac{n}{2}!\right)^2(2n)!}.$$

**9.003** If $n$ is odd the last term in the brackets is:

$$(-1)^{\frac{n-1}{2}}\frac{(n!)^2(n-1)!}{\left(\left[\frac{1}{2}(n-1)\right]!\right)^2(2n-1)!}x.$$

**9.010** If $n$ is a positive integer a second solution of Legendre's Equation is the infinite series:

$$Q_n(x) = \frac{2^n(n!)^2}{(2n+1)!}\left\{ x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}x^{-(n+5)} + \ldots . \right\}.$$

**9.011**

$$P_{2n}(\cos\theta) = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\left\{ \sin^{2n}\theta - \frac{(2n)^2}{2!}\sin^{2n-2}\theta\cos^2\theta + \ldots . + (-1)^n\frac{(2n)^2(2n-2)^2\ldots . 4^2 2^2}{(2n)!}\cos^{2n}\theta \right\}.$$

**9.012**

$$P_{2n+1}(\cos\theta) = (-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}\left\{ \sin^{2n}\theta\cos\theta - \frac{(2n)^2}{3!}\sin^{2n-2}\theta\cos^3\theta + \ldots . + (-1)^n\frac{(2n)^2(2n-2)^2\ldots . 4^2 2^2}{(2n+1)!}\cos^{2n+1}\theta \right\}.$$

(Brodetsky: Mess. of Math. 42, p. 65, 1912)

**9.02** Recurrence formulae for $P_n(x)$:

1. $$(n+1)P_{n+1} + nP_{n-1} = (2n+1)xP_n.$$

2. $$(2n+1)P_n = \frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx}.$$

3. $$(n+1)P_n = \frac{dP_{n+1}}{dx} - x\frac{dP_n}{dx}.$$

4. $$nP_n = x\frac{dP_n}{dx} - \frac{dP_{n-1}}{dx}.$$

5. $$(1-x^2)\frac{dP_n}{dx} = (n+1)(xP_n - P_{n+1}).$$

6. $$(1-x^2)\frac{dP_n}{dx} = n(P_{n-1} - xP_n).$$

7. $$(2n+1)(1-x^2)\frac{dP_n}{dx} = n(n+1)(P_{n-1} - P_{n+1}).$$

**9.028** Recurrence formulae for $Q_n(x)$. These are the same as those for $P_n(x)$.

**9.030** Special Values.

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1),$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3),$$
$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x),$$
$$P_6(x) = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5),$$
$$P_7(x) = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x),$$
$$P_8(x) = \tfrac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35).$$

**9.031**

$$Q_0(x) = \frac{1}{2}\log\frac{x+1}{x-1},$$

$$Q_1(x) = \frac{1}{2}x\log\frac{x+1}{x-1} - 1,$$

$$Q_2(x) = \frac{1}{2}P_2(x)\log\frac{x+1}{x-1} - \frac{3}{2}x,$$

$$Q_3(x) = \frac{1}{2}P_3(x)\log\frac{x+1}{x-1} - \frac{5}{2}x^2 + \frac{2}{3}.$$

**9.032**

$$P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n},$$

$$P_{2n+1}(0) = 0,$$

$$P_n(1) = 1,$$

$$P_n(-x) = (-1)^n P_n(x).$$

**9.033** If $z = r\cos\theta$:

$$\frac{\partial P_n(\cos\theta)}{\partial z} = \frac{n+1}{r}\left\{ P_1(\cos\theta)P_n(\cos\theta) - P_{n+1}(\cos\theta) \right\}$$

$$= \frac{n(n+1)}{(2n+1)r}\left\{ P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta) \right\}.$$

**9.034** Rodrigues' Formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

**9.035** If $z = r\cos\theta$:

$$P_n(\cos\theta) = \frac{(-1)^n}{n!} r^{n+1}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

**9.036** If $m \leqslant n$:

$$P_m(x)P_n(x) = \sum_{k=0}^{m} \frac{A_{m-k}A_kA_{n-k}}{A_{n+m-k}}\left(\frac{2n+2m-4k+1}{2n+2m-2k+1}\right)P_{n+m-2k}(x),$$

where:

$$A_r = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2r-1)}{r!}.$$

MEHLER'S INTEGRALS

**9.040** For all values of $n$:

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\phi - \cos\theta)}}.$$

**9.041** If $n$ is a positive integer:

$$P_n(\cos\theta) = \frac{2}{\pi}\int^\pi \frac{\sin(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\theta - \cos\phi)}}.$$

LAPLACE'S INTEGRALS, FOR ALL VALUES OF $n$

**9.042**

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \{x + \sqrt{x^2-1}\cos\phi\}^n\, d\phi.$$

**9.043**

$$Q_n(x) = \int^\infty \frac{d\phi}{\{x + \sqrt{x^2-1}\cosh\phi\}^{n+1}}.$$

### INTEGRAL PROPERTIES

**9.044**

$$\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0 \text{ if } m \neq n$$

$$= \frac{2}{2n+1} \text{ if } m = n.$$

**9.045**

$$(m-n)(m+n+1)\int_x^1 P_m(x)P_n(x)\,dx$$

$$= \tfrac{1}{2}\{P_m[(n+1)P_{n+1} - nP_{n-1}] - P_n[(m+1)P_{m+1} - mP_{m-1}]\}.$$

**9.046**

$$(2n+1)\int^1 P_n^2(x)\,dx = 1 - xP_n^2 - 2x(P_1^2 + P_2^2 + \ldots + P_{n-1}^2) + 2(P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n)$$

### EXPANSIONS IN LEGENDRE FUNCTIONS

**9.050** Neumann's expansion:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x),$$

$$a_n = (n+\tfrac{1}{2})\int_{-1}^{+1} f(x)P_n(x)\,dx,$$

$$= \frac{n+\frac{1}{2}}{2^n n!}\int_{-1}^{+1} f^{(n)}(x)\cdot(1-x^2)^n\,dx.$$

**9.051** Any polynomial in $x$ may be expressed as a series of Legendre's polynomials. If $f_n(x)$ is a polynomial of degree $n$:

$$f_n(x) = \sum_{k=0}^{n} a_k P_k(x),$$

$$a_k = \frac{2k+1}{2}\int_{-1}^{+1} f_n(x)P_k(x)\,dx.$$

### SPECIAL EXPANSIONS IN LEGENDRE FUNCTIONS

**9.060** For all positive real values of $n$:

1. $$\cos n\theta = -\frac{1+\cos n\pi}{2(n^2-1)}\left\{P_0(\cos\theta) + \frac{5n^2}{(n^2-3^2)}P_2(\cos\theta) + \frac{9n^2(n^2-2^2)}{(n^2-3^2)(n^2-5^2)}P_4(\cos\theta) + \ldots\right\} - \frac{1-\cos n\pi}{2(n^2-2^2)}\left\{3P_1(\cos\theta) + \frac{7(n^2-1^2)}{(n^2-4^2)}P_3(\cos\theta) + \frac{11(n^2-1^2)(n^2-3^2)}{(n^2-4^2)(n^2-6^2)}P_5(\cos\theta) + \ldots\right\}.$$

2. $$\sin n\theta = -\frac{1}{2}\frac{\sin n\pi}{(n^2-1)}\left\{P_0(\cos\theta) + \frac{5n^2}{(n^2-3^2)}P_2(\cos\theta) + \frac{9n^2(n^2-2^2)}{(n^2-3^2)(n^2-5^2)}P_4(\cos\theta) + \ldots\right\} + \frac{1}{2}\frac{\sin n\pi}{(n^2-2^2)}\left\{3P_1(\cos\theta) + \frac{7(n^2-1^2)}{(n^2-4^2)}P_3(\cos\theta) + \frac{11(n^2-1^2)(n^2-3^2)}{(n^2-4^2)(n^2-6^2)}P_5(\cos\theta) + \ldots\right\}.$$

**9.061** If $n$ is a positive integer:

1. $$\cos n\theta = \frac{1}{2}\frac{2\cdot4\cdot6\ldots2n}{3\cdot5\cdot7\ldots(2n+1)}\left\{(2n+1)P_n(\cos\theta) + (2n-3)\frac{[n^2-(n+1)^2]}{[n^2-(n-2)^2]}P_{n-2}(\cos\theta) + (2n-7)\frac{[n^2-(n+1)^2][n^2-(n-1)^2]}{[n^2-(n-2)^2][n^2-(n-4)^2]}P_{n-4}(\cos\theta) + \ldots\right\}.$$

2. $$\sin n\theta = \frac{\pi}{4}\frac{1\cdot3\cdot5\ldots(2n-3)}{2\cdot4\cdot6\ldots(2n-2)}\left\{(2n-1)P_{n-1}(\cos\theta) + (2n+3)\frac{[n^2-(n-1)^2]}{[n^2-(n+2)^2]}P_{n+1}(\cos\theta) + (2n+7)\frac{[n^2-(n-1)^2][n^2-(n+1)^2]}{[n^2-(n+2)^2][n^2-(n+4)^2]}P_{n+3}(\cos\theta) + \ldots\right\}.$$

**9.062**

1. $$\theta = \frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(4n-1)}{(2n-1)^2}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n-1}(\cos\theta).$$

2. $$\sin\theta = \frac{\pi}{4} - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(4n+1)}{(2n-1)(2n+2)}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n}(\cos\theta).$$

3. $$\cot\theta = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{2n(4n-1)}{(2n-1)}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n-1}(\cos\theta).$$

4. $$\csc\theta = \frac{\pi}{2} + \frac{\pi}{2}\sum_{n=1}^{\infty}(4n+1)\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n}(\cos\theta).$$

**9.063**

1. $$\log\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = 1 + \sum_{n=1}^{\infty}\frac{1}{n+1}P_n(\cos\theta).$$

2. $$\log\frac{\tan\frac{1}{4}(\pi-\theta)}{\frac{1}{2}\sin\theta} = -\log\sin\frac{\theta}{2} - \log\left(1+\sin\frac{\theta}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n}P_n(\cos\theta).$$

**9.064** $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and second kinds, and $k = \sin\theta$:

1. $$K(k) = \frac{\pi^2}{4} + \frac{\pi^2}{4}\sum_{n=1}^{\infty}(-1)^n(4n+1)\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^3 P_{2n}(\cos\theta).$$

2. $$E(k) = \frac{\pi^2}{8} + \frac{\pi^2}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(4n+1)}{(2n-1)(2n+2)}\left(\frac{1\cdot 3\cdot 5\ldots.(2n-1)}{2\cdot 4\cdot 6\ldots.2n}\right)^3 P_{2n}(\cos\theta).$$

(Hargreaves, Mess. of Math. 26, p. 89, 1897)

**9.070** The differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{ n(n+1) - \frac{m^2}{1-x^2} \right\} y = 0.$$

If $m$ is a positive integer, and $-1 > x > +1$, two solutions of this differential equation are the associated Legendre functions

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m P_n(x)}{dx^m},$$

$$Q_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m Q_n(x)}{dx^m}.$$

**9.071** If $n, m, r$ are positive integers, and $n > m$, $r > m$:

$$\int_{-1}^{+1} P_s^m(x)\, P_r^m(x)\, dx = 0 \text{ if } r \neq n,$$

$$= \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \text{ if } r = n.$$

**9.100** Bessel's Differential Equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

**9.101** One solution is:

$$y = J_\nu(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{\nu+2k}}{2^{\nu+2k}k!\Gamma(\nu+k+1)}.$$

**9.102** A second independent solution when $\nu$ is not an integer is:

$$y = J_{-\nu}(x).$$

**9.103** If $\nu = n$, an integer:

$$J_{-n}(x) = (-1)^n J_n(x).$$

**9.104** A second independent solution when $\nu = n$, an integer, is:

$$\pi Y_n(x) = 2J_n(x)\cdot\log\frac{x}{2} - \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$

$$- \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}\left\{ \psi(k+1) + \psi(k+n+1) \right\}$$

(see **6.61**).

**9.105** For all values of $\nu$, whether integral or not:

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\Big(\cos \nu\pi J_\nu(x) - J_{-\nu}(x)\Big),$$

$$J_{-\nu}(x) = \cos \nu\pi J_\nu(x) - \sin \nu\pi Y_\nu(x),$$

$$Y_{-\nu}(x) = \sin \nu\pi J_\nu(x) + \cos \nu\pi Y_\nu(x).$$

**9.106** For $\nu = n$, an integer:

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

**9.107** Cylinder Functions of the third kind, solutions of Bessel's differential equation:

1. $$H_\nu^{\mathrm{I}}(x) = J_\nu(x) + iY_\nu(x).$$

2. $$H_\nu^{\mathrm{II}}(x) = J_\nu(x) - iY_\nu(x).$$

3. $$H_{-\nu}^{\mathrm{I}}(x) = e^{\nu\pi i} H_\nu^{\mathrm{I}}(x).$$

4. $$H_{-\nu}^{\mathrm{II}}(x) = e^{-\nu\pi i} H_\nu^{\mathrm{II}}(x).$$

**9.110** Recurrence formulae satisfied by the functions $J_\nu$, $Y_\nu$, $H_\nu^{\mathrm{I}}$, $H_\nu^{\mathrm{II}}$, $C_\nu$ represents any one of these functions.

1. $$C_{\nu-1}(x) - C_{\nu+1}(x) = 2\frac{d}{dx}C_\nu(x).$$

2. $$C_{-1}(x) + C_{\nu+1}(x) = \frac{2\nu}{x}C_\nu(x).$$

3. $$\frac{d}{dx}C_\nu(x) = C_{\nu-1}(x) - \frac{\nu}{x}C_\nu(x).$$

4. $$\frac{d}{dx}C(x) = \frac{\nu}{x}C_\nu(x) - C_{\nu+1}(x).$$

5. $$\frac{d}{dx}\Big\{x^\nu C_{\nu(x)}\Big\} = x^\nu C_{\nu-1}(x).$$

6. $$\frac{d^2C_\nu(x)}{dx^2} = \frac{1}{4}\Big\{C_{\nu+2}(x) + C_{\nu-2}(x) - 2C_\nu(x)\Big\}.$$

**9.111**

1. $J_\nu(x)\dfrac{dY_\nu(x)}{dx} - Y_\nu(x)\dfrac{dJ_\nu(x)}{dx} = \dfrac{2}{\pi x}.$ 2. $J_{\nu+1}(x)Y_\nu(x) - J_\nu(x)Y_{\nu+1}(x) = \dfrac{2}{\pi x}.$

ASYMPTOTIC EXPANSIONS FOR LARGE VALUES OF $x$

**9.120**

1. $$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{P(x)\cos\left(x - \frac{2\nu+1}{4}\pi\right) - Q_\nu(x)\sin\left(x - \frac{2\nu+1}{4}\pi\right)\right\},$$

2. $$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{P_\nu(x)\sin\left(x - \frac{2\nu+1}{4}\pi\right) + Q_\nu(x)\cos\left(x - \frac{2\nu+1}{4}\pi\right)\right\},$$

3. $H_\nu^{(1)}(x) = e^{i\left(x-\frac{2\nu+1}{4}\pi\right)}\sqrt{\frac{2}{\pi x}}\left\{P_\nu(x) + iQ_\nu(x)\right\}$,

4. $H_\nu^{(2)}(x) = e^{-i\left(x-\frac{2\nu+1}{4}\pi\right)}\sqrt{\frac{2}{\pi x}}\left\{P_\nu(x) - iQ_\nu(x)\right\}$,

where

$$P_\nu(x) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{(4\nu^2-1^2)(4\nu^2-3^2)\ldots\ldots(4\nu^2-\overline{4k-1}^2)}{(2k)!\ 2^{6k}\ x^{2k}},$$

$$Q_\nu(x) = \sum_{k=1}^{\infty}(-1)^{k+1} \frac{(4\nu^2-1^2)(4\nu^2-3^2)\ldots\ldots(4\nu^2-\overline{4k-3}^2)}{(2k-1)!\ 2^{6k-3}\ x^{2k-1}}.$$

SPECIAL VALUES

**9.130**

1. $J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots.$

2. $J_1(x) = -\frac{dJ_0(x)}{dx} = \frac{x}{2}\left\{1 - \frac{1}{1!2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^6 + \ldots\right\}.$

3. $$\frac{\pi}{2}Y_0(x) = \left(\log\frac{x}{2}+\gamma\right)J_0(x) + \left(\frac{x}{2}\right)^2 - \frac{1}{(2!)^2}\left(1+\frac{1}{2}\right)\left(\frac{x}{2}\right)^4 + \frac{1}{(3!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{x}{2}\right)^6 - \ldots$$
$$= \left(\log\frac{x}{2}+\gamma\right)J_0(x) + 4\left\{\frac{1}{2}J_2(x) - \frac{1}{4}J_4(x) + \frac{1}{6}J_6(x) - \ldots\right\}.$$

4. $$\frac{\pi}{2}Y_1(x) = \left(\log\frac{x}{2}+\gamma\right)J_1(x) - \frac{1}{x}J_0(x) - \frac{x}{2}\left\{1 - \frac{1}{1!2!}\left(1+\frac{1}{2}\right)\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{x}{2}\right)^4 - \ldots.\right\}$$
$$= \left(\log\frac{x}{2}+\gamma\right)J_1(x) - \frac{1}{x}J_0(x) + \frac{3}{1\cdot 2}J_3(x) - \frac{5}{2\cdot 3}J_5(x) + \frac{7}{3\cdot 4}J_7(x) - \ldots.$$

$\gamma = 0.5772157$ (6.602).

**9.131** Limiting values for $x = 0$:

$$J_0(x) = 1,$$
$$J_1(x) = 0,$$
$$Y_0(x) = \frac{2}{\pi}\left(\log\frac{x}{2}+\gamma\right),$$
$$Y_1(x) = -\frac{2}{\pi x}.$$

**9.132** Limiting values for $x = \infty$:

$$J_0(x) = \frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}}, \qquad Y_0(x) = \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}},$$

$$J_1(x) = \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}}, \qquad Y_1(x) = -\frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}}.$$

**9.140** Bessel's Addition Formula:

$$J_\nu(x + h) = \left(\frac{x + h}{x}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{h^k}{k!} \left(\frac{2x + h}{2x}\right)^k J_{\nu+k}(x).$$

**9.141** Multiplication formula:

$$J_\nu(\alpha x) = \alpha^\nu \sum_{k=0}^{\infty} \frac{(1 - \alpha^2)^k}{k!} \left(\frac{x}{2}\right)^k J_{\nu+k}(x).$$

**9.142**

$$J_\nu(\alpha x) J_\mu(\beta x) = \sum_{k=0}^{\infty} (-1)^k A_k \left(\frac{x}{2}\right)^{\mu+\nu+2k},$$

where

$$A_k = \sum_{s=0}^{k} \frac{\alpha^{2k-2s}\beta^{2s}}{s!(k-s)!\Gamma(\nu + k - s + 1)\Gamma(\mu + s + 1)}.$$

**9.143**

$$J_\nu(x) J_\mu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1)\Gamma(\mu + k + 1)} \binom{\mu + \nu + 2k}{k} \left(\frac{x}{2}\right)^{\mu+\nu+2k}.$$

## DEFINITE INTEGRAL EXPRESSIONS FOR BESSEL'S FUNCTIONS

**9.150**

$$J_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos(x \sin\phi) \cos^{2\nu}\phi \cdot d\phi.$$

**9.151**

$$J_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi} \cos(x \cos\phi) \sin^{2\nu}\phi \cdot d\phi.$$

**9.152**

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\pi e^{ix\cos\phi} \sin^{2\nu}\phi \cdot d\phi.$$

If $n$ is an integer:

**9.153**

$$J_{2n}(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)\cos(2n\phi)d\phi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}.$$

**9.154**

$$J_{2n}(x) = \frac{(-1)^n}{\pi}\int_0^\pi \cos(x\cos\phi)\cos(2n\phi)d\phi = \frac{2(-1)^n}{\pi}\int_0^{\frac{\pi}{2}}.$$

**9.155**

$$J_{2n+1}(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\phi)\sin(2n+1)\,\phi\,d\phi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}$$

**9.156**

$$J_{2n+1}(x) = \frac{(-1)^n}{\pi}\int_0^\pi \sin(x\cos\phi)\cos(2n+1)\phi d\phi = \frac{2(-1)^n}{\pi}\int_0^{\frac{\pi}{2}}.$$

**9.157**

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-in\phi+ix\sin\phi}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi+ix\sin\phi}\,d\phi.$$

## INTEGRAL PROPERTIES

**9.160** If $C_\nu(\mu x)$ is any one of the particular integrals:

$$J_\nu(\mu x),\ Y_\nu(\mu x),\ H_\nu^{\mathrm{I}}(\mu x),\ H_\nu^{\mathrm{II}}(\mu x),$$

of the differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\mu^2 - \frac{\nu^2}{x^2}\right)y = 0,$$

$$\int_a^b C_\nu(\mu_k x)C_\nu(\mu_l x)x dx$$

$$= \frac{1}{\mu_k^2 - \mu_l^2}\left[x\left\{\mu_l C_\nu(\mu_k x)C_\nu'(\mu_l x) - \mu_k C_\nu(\mu_l x)C_\nu'(\mu_k x)\right\}\right]_a^b;\ \mu_k \neq \mu_l.$$

**9.161** If $\mu_k$ and $\mu_l$ are two different roots of

$$C_\nu(\mu b) = 0,$$

$$\int_a^b C_\nu(\mu_k x)C_\nu(\mu_l x)x\,dx = \frac{a}{\mu_k^2 - \mu_l^2}\left\{\mu_k C_\nu(\mu_l a)C_\nu'(\mu_k a) - \mu_l C_\nu(\mu_k a)C_\nu'(\mu_l a)\right\}.$$

**9.162** If $\mu_k$ and $\mu_l$ are two different roots of

$$a\frac{C_\nu'(\mu a)}{C_\nu(\mu a)} = p\mu + q\frac{1}{\mu},$$

and

$$C_\nu(\mu b) = 0,$$

$$\int_b C_\nu(\mu_k x)C_\nu(\mu_l x)x dx = pC_\nu(\mu_k a)C_\nu(\mu_l a).$$

If $\mu_k = \mu_l$:

$$\int_b C_\nu(\mu_k x)C_\nu(\mu_l x)x dx = \frac{1}{2}\left\{b^2 C_\nu'^2(\mu_k b) - a^2 C_\nu'^2(\mu_k a) - \left(a^2 - \frac{\nu^2}{\mu_k^2}\right)C_\nu^2(\mu_k a)\right\}.$$

## EXPANSIONS IN BESSEL'S FUNCTIONS

**9.170** Schlömilch's Expansion. Any function $f(x)$ which has a continuous differential coefficient for all values of $x$ in the closed range $(0, \pi)$ may be expanded in the series:

$$f(x) = a_0 + \sum_{k=1} a_k J_0(kx),$$

where

$$a_0 = f(0) + \frac{1}{\pi}\int_0^{\pi} u \int_0^{\frac{\pi}{2}} f'(u \sin\theta)\, d\theta\, du,$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} u \cos ku \int_0^{\frac{\pi}{2}} f'(u \sin\theta)\, d\theta\, du.$$

**9.171**

$$f(x) = a_0 x^n + \sum_{k=1}^{\infty} a_k J_n(\alpha_k x) \qquad 0<x<1,$$

where

$$J_{n+1}(\alpha_k) = 0,$$

$$a_0 = 2(n+1)\int_0^1 f(x) x^{n+1}\, dx,$$

$$a_k = \frac{2}{[J_n(\alpha_k)]^2}\int_0^1 x f(x) J_n(\alpha_k x)\, dx.$$

(Bridgman, Phil. Mag. 16, p. 947, 1908)

**9.172**

$$f(x) = \sum_{k=1}^{\infty} A_k J_0(\mu_k x) \qquad a<x<b,$$

where:

$$a\frac{J_0'(\mu_k a)}{J_0(\mu_k a)} = p\mu_k + \frac{q}{\mu_k},$$

and

$$J_0(\mu_k b) = 0,$$

$$A_k = 2\,\frac{\int_a^b x f(x) J_0(\mu_k x)\, dx - p f(a) J_0(\mu_k a)}{b^2 J_0'^2(\mu_k b) - a^2 J_0'^2(\mu_k a) - (a^2 + 2p) J_0^2(\mu_k a)}.$$

(Stephenson, Phil. Mag. 14, p. 547, 1907)

## SPECIAL EXPANSIONS IN BESSEL'S FUNCTIONS

**9.180**

1. $\sin x = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(x),$

2. $\cos x = J_0(x) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(x).$

**9.181**

1. $$\cos(x\sin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos 2k\theta,$$

2. $$\sin(x\sin\theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(x)\sin(2k+1)\theta.$$

**9.182**

1. $$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} \frac{(n+2k)\,(n+k-1)!}{k!} J_{n+2k}(x),$$

2. $$\sqrt{\frac{2x}{\pi}} = \sum_{k=0}^{\infty} \frac{(4k+1)\,(2k)!}{2^{2k}(k!)^2} J_{2k+\frac{1}{2}}(x).$$

**9.183**

$$\frac{dJ_\nu(x)}{d\nu} = \left\{\log\frac{x}{2} - \psi(\nu+1)\right\} J(x) + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{\nu+2k}{k(\nu+k)} J_{\nu+2k}(x)$$

$$= J_\nu(x)\log\frac{x}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{\psi(\nu+k+1)}{k!\;\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k}. \qquad \text{(see 6.61)}$$

**9.200** The differential equation:

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \left(\mu^2 - \frac{n(n+1)}{x^2}\right)y = 0$$

with the substitution:

$$z = y\sqrt{x}, \qquad \mu x = \rho$$

becomes:

$$\frac{d^2z}{d\rho^2} + \frac{1}{\rho}\frac{dz}{d\rho} + \left(1 - \frac{(n+\frac{1}{2})^2}{\rho^2}\right)z = 0$$

which is Bessel's equation of order $n + \frac{1}{2}$.

**9.201** Two independent solutions are:

$$z = J_{n+\frac{1}{2}}(\rho).$$

$$z = J_{-n-\frac{1}{2}}(\rho).$$

The former remains finite for $\rho = 0$; the latter becomes infinite for $\rho = 0$.

**9.202** Special values.

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J\ (x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right),$$

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left(\frac{3}{x^2} - 1\right) \sin x - \frac{3}{x} \cos x \right\},$$

$$J_{\frac{7}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left(\frac{15}{x^3} - \frac{6}{x}\right) \sin x - \left(\frac{15}{x^2} - 1\right) \cos x \right\},$$

$$J_{\frac{9}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right) \sin x - \left(\frac{105}{x^3} - \frac{10}{x}\right) \cos x \right\}.$$

**9.203**

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

$$J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \left(\sin x + \frac{\cos x}{x}\right),$$

$$J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \frac{3}{x} \sin x + \left(\frac{3}{x^2} - 1\right) \cos x \right\},$$

$$J_{-\frac{7}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \left\{ \left(\frac{15}{x^2} - 1\right) \sin x + \left(\frac{15}{x^3} - \frac{6}{x}\right) \cos x \right\},$$

$$J_{-\frac{9}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left(\frac{105}{x^3} - \frac{10}{x}\right) \sin x + \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right) \cos x \right\}.$$

**9.204**

$$H^{\mathrm{I}}_{\frac{1}{2}}(x) = -i\sqrt{\frac{2}{\pi x}} e^{ix},$$

$$H^{\mathrm{I}}_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}} e^{ix} \left(1 + \frac{i}{x}\right),$$

$$H^{\mathrm{I}}_{\frac{5}{2}}(x) = -\sqrt{\frac{2}{\pi x}} e^{ix} \left\{ \frac{3}{x} + i\left(\frac{3}{x^2} - 1\right) \right\}.$$

**9.205**

$$H^{\mathrm{II}}_{\frac{1}{2}}(x) = i\sqrt{\frac{2}{\pi x}} e^{-ix},$$

$$H^{\mathrm{II}}_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}} e^{-ix} \left(1 - \frac{i}{x}\right),$$

$$H^{\mathrm{II}}_{\frac{5}{2}}(x) = -\sqrt{\frac{2}{\pi x}} e^{-ix} \left\{ \frac{3}{x} - i\left(\frac{3}{x^2} - 1\right) \right\}.$$

**9.210** The differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) y = 0,$$

with the substitution,

$$x = iz,$$

becomes Bessel's equation.

**9.211** Two independent solutions of **9.210** are:

$$I_\nu(x) = i^{-\nu} J_\nu(ix),$$

$$K^\nu(x) = e^{\frac{\nu+1}{2}\pi i} \frac{\pi}{2} H_\nu^{1}(ix).$$

**9.212** If $\nu = n$, an integer:

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{n+2k},$$

$$K_n(x) = i^{n+1} \frac{\pi}{2} H_n^{I}(x).$$

**9.213**

$$I_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}\left(\frac{x}{2}\right)^\nu \int_0^\pi \cosh(x\cos\phi)\sin^{2\nu}\phi\, d\phi,$$

$$K_\nu(x) = \frac{\sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})}\left(\frac{x}{2}\right)^\nu \int^\infty \sinh^{2\nu}\phi\, e^{-x\cosh\phi}\, d\phi.$$

**9.214** If $x$ is large, to a first approximation:

$$I_n(x) = (2\pi x\cosh\beta)^{-\frac{1}{2}} e^{x(\cosh\beta - \beta\sinh\beta)},$$

$$K_n(x) = \pi(2\pi x\cosh\beta)^{-\frac{1}{2}} e^{-x(\cosh\beta - \beta\sinh\beta)},$$

$$n = x\sinh\beta.$$

**9.215** Ber and Bei Functions.

$$\operatorname{ber} x + i \operatorname{bei} x = I(x\sqrt{i}),$$

$$\operatorname{ber} x - i \operatorname{bei} x = I_0(ix\sqrt{i}),$$

$$\operatorname{ber} x = 1 - \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \frac{1}{(4!)^2}\left(\frac{x}{2}\right)^8 - \cdots$$

$$\operatorname{bei} x = \left(\frac{x}{2}\right)^2 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \frac{1}{(5!)^2}\left(\frac{x}{2}\right)^{10} - \cdots$$

**9.216** Ker and Kei Functions:

$$\ker x + i \operatorname{kei} x = K_0(x\sqrt{i}),$$

$$\ker x - i \operatorname{kei} x = K_0(ix\sqrt{i}),$$

$$\ker x = \left(\log \frac{2}{x} - \gamma\right) \operatorname{ber} x + \frac{\pi}{4} \operatorname{bei} x - \frac{1}{(2!)^2}\left(1 + \frac{1}{2}\right)\left(\frac{x}{2}\right)^4 + \frac{1}{(4!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)\left(\frac{x}{2}\right)^8 - \cdots$$

$$\operatorname{kei} x = \left(\log \frac{2}{x} - \gamma\right) \operatorname{bei} x - \frac{\pi}{4} \operatorname{ber} x + \left(\frac{x}{2}\right)^2 - \frac{1}{(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\left(\frac{x}{2}\right)^6 + \cdots.$$

**9.220** The Bessel-Clifford Differential Equation:

$$x\frac{d^2y}{dx^2} + (\nu + 1)\frac{dy}{dx} + y = 0.$$

With the substitution:

$$z = x^{\nu/2}y \qquad u = 2\sqrt{x},$$

the differential equation reduces to Bessel's equation.

**9.221** Two independent solutions of **9.220** are:

$$C_\nu(x) = x^{-\frac{\nu}{2}} J_\nu\,(2\sqrt{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!\,\Gamma(\nu + k + 1)},$$

$$D_\nu(x) = x^{-\frac{\nu}{2}}\, Y_\nu(2\sqrt{x}).$$

**9.222**

$$C_{\nu+1}(x) = -\frac{d}{dx}\, C_\nu(x),$$

$$xC_{\nu+2}(x) = (\nu + 1)C_{\nu+1}(x) - C_\nu(x).$$

**9.223** If $\nu = n$, an integer:

$$C_n(x) = (-1)^n \frac{d^n}{dx^n}\, C_0(x),$$

$$C_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(k!)^2}$$

**9.224** Changing the sign of $\nu$, the corresponding solution of:

$$x\frac{d^2y}{dx^2} - (\nu - 1)\frac{dy}{dx} + y = 0,$$

$$y = x^\nu C_\nu(x).$$

**9.225** If $\nu$ is half an odd integer:

$$C_{\frac{1}{2}}(x) = \frac{\sin(2\sqrt{x} + \epsilon)}{2\sqrt{x}},$$

$$C_{\frac{3}{2}}(x) = -\frac{d}{dx}C_{\frac{1}{2}}(x) = \frac{\sin(2\sqrt{x} + \epsilon)}{4x^{\frac{3}{2}}} - \frac{\cos(2\sqrt{x} + \epsilon)}{2x},$$

$$C_{\frac{5}{2}}(x) = -\frac{d}{dx}C_{\frac{3}{2}}(x) = \frac{3 - 4x}{8x^{\frac{5}{2}}}\sin(2\sqrt{x} + \epsilon) - \frac{3\cos(2\sqrt{x} + \epsilon)}{4x^2},$$

. . . . .

. . . . .

$$C_{-\frac{1}{2}}(x) = -\cos(2\sqrt{x} + \epsilon),$$

$$C_{-\frac{3}{2}}(x) = x^3 C_{\frac{3}{2}}(x),$$

$$C_{-\frac{5}{2}}(x) = x^5 C_{\frac{5}{2}}(x).$$

. . .

. . .

$\epsilon$ is arbitrary so as to give a second arbitrary constant.

**9.226** For $x$ negative, the solution of the equation:

$$x\frac{d^2y}{dx^2} + (\pm\nu + 1)\frac{dy}{dx} - y = 0,$$

when $\nu$ is half an odd integer, is obtained from the values in **9.225** by changing sin and cos to sinh and cosh respectively.

**9.227**

$$(m + n + 1)\int C_{m+1}(x)C_{n+1}(x)\,dx = -xC_{m+1}(x)C_{n+1}(x) - C_m(x)C_n(x),$$

$$(m + n + 1)\int x^{m+n}C_m(x)C_n(x)\,dx = x^{m+n+1}\left\{xC_{m+1}(x)C_{n+1}(x) + C_m(x)C_n(x)\right\}.$$

**9.228**

1. $$\int_0^{\pi} C_{-\frac{1}{2}}(x\cos^2\phi)\,d\phi = \pi C_0(x).$$

2. $$\int_0^{\pi} C_{\frac{1}{2}}(x\cos^2\phi)\,d\phi = \pi C_1(x).$$

3. $$\int_0^{\pi} C_0(x\sin^2\phi)\sin\phi\,d\phi = C_{\frac{1}{2}}(x).$$

4. $$\int_0^{\pi} C_1(x\sin^2\phi)\sin^3\phi\,d\phi = C_{\frac{3}{2}}(x).$$

5. $$\int_0^{\pi} C_1(x\sin^2\phi)\sin\phi\,d\phi = \frac{1 - \cos 2\sqrt{x}}{x}.$$

**9.229** Many differential equations can be solved in a simpler form by the use of the $C_n$ functions than by the use of Bessel's functions.

(Greenhill, Phil. Mag. 38, p. 501, 1919)

**9.240** The differential equation:

$$\frac{d^2y}{dx^2} + \frac{2(n+1)}{x}\frac{dy}{dx} + y = 0,$$

with the change of variable:

$$y = zx^{-n-\frac{1}{2}},$$

becomes Bessel's equation **9.200.**

**9.241** Solutions of **9.240** are:

1. $y = x^{-n-\frac{1}{2}}\, J_{n+\frac{1}{2}}(x).$
2. $y = x^{-n-\frac{1}{2}}\, Y_{n+\frac{1}{2}}(x).$
3. $y = x^{-n-\frac{1}{2}}\, H^{\mathrm{I}}_{n+\frac{1}{2}}(x).$
4. $y = x^{-n-\frac{1}{2}}\, H^{\mathrm{II}}_{n+\frac{1}{2}}(x).$

**9.242** The change of variable:

$$x = 2\sqrt{z},$$

transforms equation **9.240** into the Bessel-Clifford differential equation **9.220.** This leads to a general solution of **9.240:**

$$y = C_{n+\frac{1}{2}}\left(\frac{x^2}{4}\right).$$

When $n$ is an integer the equations of **9.225** may be employed.

$$C_1\left(\frac{x^2}{4}\right) = \frac{\sin(x+\epsilon)}{x},$$

$$C_{\frac{3}{2}}\left(\frac{x^2}{4}\right) = \frac{2\sin(x+\epsilon)}{x^3} - \frac{\cos(x+\epsilon)}{x}.$$

. . .

. . .

**9.243** The solution of

$$\frac{d^2y}{dx^2} + \frac{2(n+1)}{x}\frac{dy}{dx} - y = 0,$$

may be obtained from **9.242** by writing sinh and cosh for sin and cos respectively.

**9.244** The differential equation **9.240** is also satisfied by the two independent functions (when $n$ is an integer):

$$\psi_n(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}$$

$$= \frac{1}{1\cdot 3\cdot 5\,\ldots\,(2n+1)} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^k k!\,(2n+3)\,\ldots\,\ldots\,(2n+2k+1)},$$

$$\Psi_n(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x}$$

$$= \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{x^{2n+1}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^k k!(1-2n)(3-2n)\ldots(2k-2n-1)}.$$

**9.245** The general solution of **9.240** may be written:

$$y = \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{Ae^{ix} + Be^{-ix}}{x}.$$

**9.246** Another particular solution of **9.240** is:

$$y = f_n(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{e^{-ix}}{x} = \Psi_n(x) - i\psi_n(x),$$

$$f_n(x) = \frac{i^n e^{-ix}}{x^{n+1}}\left\{1 + \frac{n(n+1)}{2ix} + \frac{(n-1)n(n+1)(n+2)}{2\cdot 4\cdot (ix)^2} + \ldots + \frac{1\cdot 2\cdot 3\ldots 2n}{2\cdot 4\cdot 6\ldots 2n(ix)^n}\right\}.$$

**9.247** The functions $\psi_n(x)$, $\Psi_n(x)$, $f_n(x)$ satisfy the same recurrence formulae:

$$\frac{d\psi_n(x)}{dx} = -x\psi_{n+1}(x),$$

$$x\frac{d\psi_n(x)}{dx} + (2n+1)\psi_n(x) = \psi_{n-1}(x).$$

**9.260** The differential equation:

$$\frac{d^2y}{dx^2} - \frac{n(n+1)}{x^2}y + y = 0,$$

with the change of variable:

$$y = u\sqrt{x}$$

is transformed into Bessel's equation of order $n + \frac{1}{2}$.

**9.261** Solutions of **9.260** are:

1. $$S_n(x) = \sqrt{\frac{\pi x}{2}}\, J_{n+\frac{1}{2}}(x) = x^{n+1}\left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}.$$

2. $$C_n(x) = (-1)^n\sqrt{\frac{\pi x}{2}}\, J_{-n-\frac{1}{2}}(x) = x^{n+1}\left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x}.$$

3. $$E_n(x) = C_n(x) - iS_n(x) = x^{n+1}\left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{e^{-ix}}{x}.$$

**9.262** The functions $S_n(x)$, $C_n(x)$, $E_n(x)$ satisfy the same recurrence formulae:

1. $$\frac{dS_n(x)}{dx} = \frac{n+1}{x}S_n(x) - S_{n+1}(x).$$

2. $$\frac{dS_n(x)}{dx} = S_{n-1}(x) - \frac{n}{x} S_n(x).$$

3. $$S_{n+1}(x) = \frac{2n+1}{x} S_n(x) - S_{n-1}(x).$$

**9.30** The hypergeometric differential equation:

$$x(1-x)\frac{d^2y}{dx^2} + \left\{ \gamma - (\alpha+\beta+1)x \right\} \frac{dy}{dx} - \alpha\beta y = 0.$$

**9.31** The equation **9.30** is satisfied by the hypergeometric series:

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha}{1}\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{1\cdot 2}\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1\cdot 2\cdot 3}\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots\cdots$$

The series converges absolutely when $x<1$ and diverges when $x>1$. When $x = +1$ it converges only when $\alpha+\beta-\gamma<0$, and then absolutely. When $x = -1$ it converges only when $\alpha+\beta-\gamma-1<0$, and absolutely if $\alpha+\beta-\gamma<0$.

**9.32**

$$\frac{d}{dx}F(\alpha, \beta, \gamma, x) = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1, \gamma+1, x).$$

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

**9.33** Representation of various functions by hypergeometric series.

$$(1+x)^n = F(-n, \beta, \beta, -x),$$

$$\log(1+x) = xF(1, 1, 2, -x),$$

$$e^x = \operatorname*{Limit}_{\beta=\infty} F\left(1, \beta, 1, \frac{x}{\beta}\right),$$

$$(1+x)^n + (1-x)^n = 2F\left(-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \frac{1}{2}, x^2\right),$$

$$\log\frac{1+x}{1-x} = 2xF\left(\frac{1}{2}, 1, \frac{3}{2}, x^2\right),$$

$$\cos nx = F\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, \sin^2 x\right),$$

$$\sin nx = n\sin xF\left(\frac{n+1}{2}, \frac{1-n}{2}, \frac{3}{2}, \sin^2 x\right),$$

$$\cosh x = \underset{\alpha=\beta=\infty}{\text{Limit}} F\left(\alpha, \beta, \frac{1}{2}, \frac{x^2}{4\alpha\beta}\right),$$

$$\sin^{-1} x = xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right),$$

$$\tan^{-1} x = xF\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right),$$

$$P_n(x) = F\left(-n, n+1, 1, \frac{1-x}{2}\right),$$

$$Q_n(x) = \frac{\sqrt{\pi}\Gamma(n+1)}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)}\frac{1}{x^{n+1}}F\left(\frac{n+1}{2}, \frac{n+2}{2}, n+\frac{3}{2}, \frac{1}{x^2}\right).$$

**9.4** Heaviside's Operational Methods of Solving Partial Differential Equations.

**9.41** The partial differential equation,

$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

where $a$ is a constant, may be solved by Heaviside's operational method.

Writing $\frac{\partial}{\partial t} = p$, and $\frac{p}{a} = q^2$, the equation becomes,

$$\frac{\partial^2 u}{\partial x^2} = q^2 u,$$

whose complete solution is $u = e^{qx}A + e^{-qx}B$, where $A$ and $B$ are integration constants to be determined by the boundary conditions. In many applications the solution $u = e^{-qx}B$, only, is required: and the boundary conditions will lead to $u = e^{-qx}f(q)u_0$, where $u_0$ is a constant. If $e^{-qx}f(q)$ be expanded in an infinite power series in $q$, and the integral and fractional, positive and negative powers of $p$ be interpreted as in **9.42,** the resulting series will be a solution of the differential equation, satisfying the boundary conditions, and reducing to $u = 0$ at $t = 0$. The expansion of $e^{-qx}f(q)$ may be carried out in two or more ways, leading to series suitable for numerical calculation under different conditions.

**9.42** Fractional Differentiation and Integration.

In the following expressions, 1 stands for a function of $t$ which is zero up to $t = 0$, and equal to 1 for $t > 0$.

**9.421**

$$p^{\frac{1}{2}}1 = \frac{1}{\sqrt{\pi t}}$$

$$p^{\frac{3}{2}}1 = \frac{1}{2t\sqrt{\pi t}} \qquad p^{\frac{2n+1}{2}}1 = (-1)^n \frac{1\cdot 3\cdot 5 \cdot \cdots (2n-1)}{2^n t^n \sqrt{\pi t}}$$

$$p^{\frac{5}{2}}1 = \frac{3}{2^2 t^2 \sqrt{\pi t}}$$

. . .

. . .

**9.422**

$$p\,1 = 0$$
$$p^2 1 = 0 \qquad p^n 1 = 0$$
$$p^3 1 = 0$$

. . .

. . .

**9.423**

$$p^{-\frac{1}{2}} = 2\sqrt{\frac{t}{\pi}}$$

$$p^{-\frac{3}{2}} = \frac{2^2 t}{3}\sqrt{\frac{t}{\pi}} \qquad p^{-\frac{2n+1}{2}}1 = \frac{2^{2n-1}t^n}{1\cdot 3\cdot 5 \cdot \cdots (2n+1)}\sqrt{\frac{t}{\pi}}$$

$$p^{-\frac{5}{2}} = \frac{2^3 t^2}{3\cdot 5}\sqrt{\frac{t}{\pi}}$$

. . .

. . .

**9.424**

$$\frac{1}{p^\nu} = \frac{t^\nu}{\Gamma(1+\nu)},$$

where $\nu$ may have any real value, except a negative integer. (Conjectural.)

**9.425**

$$\frac{p}{p-a}1 = e^{at}$$

$$\frac{1}{p-a}1 = \frac{1}{a}(e^{at} - 1)$$

**9.426** With $p = aq^2$,

$$q^{2n+1}1 = (-1)^n \frac{1\cdot 3\cdot 5 \cdot \cdots (2n-1)}{(2at)^n \sqrt{\pi a t}}$$

$$q^{-2n}1 = \frac{(at)^n}{n!}.$$

**9.427**

$$qe^{-qx}\mathrm{I} = \frac{\mathrm{I}}{\sqrt{\pi at}}\, e^{-\frac{x^2}{4at}}$$

**9.428** If $z = \dfrac{x}{2\sqrt{at}}$,

$$e^{-qx} = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-v^2} dv$$

$$\frac{\mathrm{I}}{q}\, e^{-qx} = \frac{x}{\sqrt{\pi}} \int_z^\infty e^{-v^2} \frac{dv}{v^2}.$$

**9.43** Many examples of the use of this method are given by Heaviside: Electromagnetic Theory, Vol. II. Bromwich, Proceedings Cambridge Philosophical Society, XX, p. 411, 1921, has justified its application by the method of contour integration and applied it to the solution of a problem in the conduction of heat.

**9.431** Herlitz, Arkiv for Matematik, Astronomi och Fysik, XIV, 1919, has shown that the same methods may be applied to the more general partial differential equations of the type,

$$\sum_{\alpha,\beta} A_{\alpha,\beta}(x) \frac{\partial^{\alpha+\beta}(u)}{\partial x^\alpha\, \partial t^\beta} = 0,$$

and the relations of **9.42** are valid.

**9.44** Heaviside's Expansion Theorem.

The operational solution of the differential equation of **9.41,** or the more general equation, **9.431,** satisfying the given boundary conditions, may be written in the form,

$$u = \frac{F(p)}{\Delta(p)}\, u_0,$$

where $F(p)$ and $\Delta(p)$ are known functions of $p = \dfrac{\partial}{\partial t}$. Then Heaviside's Expansion Theorem is:

$$u = u_0 \left\{ \frac{F(0)}{\Delta(0)} + \sum \frac{F(\alpha)}{\alpha \Delta'(\alpha)}\, e^{\alpha t} \right\},$$

where $\alpha$ is any root, except 0, of $\Delta(p) = 0$, $\Delta'(p)$ denotes the first derivative of $\Delta(p)$ with respect to $p$, and the summation is to be taken over all the roots of $\Delta(p) = 0$. This solution reduces to $u = 0$ at $t = 0$.

Many applications of this expansion theorem are given by Heaviside, Electromagnetic Theory, II, and III; Electrical Papers, Vol. II. Herlitz, **9.431,** has also applied this expansion theorem to the solution of the problem of the distribution of magnetic induction in cylinders and plates.

**9.45** Bromwich's Expansion Theorem. Bromwich has extended Heaviside's Expansion Theorem as follows. If the operational solution of the partial differential equation of **9.41,** obtained to satisfy the boundary conditions, is

$$u = \frac{F(p)}{\Delta(p)}\, (Gt)$$

where $G$ is a constant, then the solution of the differential equation is

$$u = G\left\{ N_0 t + N_1 + \sum \frac{F(\alpha)}{\alpha^2 \Delta'(\alpha)} e^{\alpha t} \right\},$$

where $N_0$ and $N_1$ are defined by the expansion,

$$\frac{F(p)}{\Delta(p)} = N_0 + N_1 p + N_2 p^2 + \ldots;$$

$\alpha$ is any root of $\Delta(p) = 0$, $\Delta'(p)$ is the first derivative of $\Delta(p)$ with respect to $p$, and the summation is over all the roots, $\alpha$. This solution reduces to $u = 0$ at $t = 0$. Phil. Mag. 37, p. 407, 1919; Proceedings London Mathematical Society, 15, p. 401, 1916.

**9.9** References to Bessel Functions.

Nielsen: Handbuch der Theorie der Cylinder Funktionen.

Leipzig, 1904.

The notation and definitions given by Nielsen have been adopted in the present collection of formulae. The only difference is that Nielsen uses an upper index, $J^n(x)$, to denote the order, where the more usual custom of writing $J_n(x)$ is here employed. In place of $H_1{}^n$ and $H_2{}^n$ used by Nielsen for the cylinder functions of the third kind, $H_n{}^{\mathrm{I}}$ and $H_n{}^{\mathrm{II}}$ are employed in this collection.

Gray and Mathews: Treatise on Bessel Functions.

London, 1895.[1]

The Bessel Function of the second kind, $Y_n(x)$, employed by Gray and Mathews is the function

$$\frac{\pi}{2} Y_n(x) + (\log 2 - \gamma) J_n(x),$$

of Nielsen.

Schafheitlin: Die Theorie der Besselschen Funktionen.

Leipzig, 1908.

Schafheitlin defines the function of the second kind, $Y_n(x)$, in the same way as Nielsen, except that its sign is changed.

NOTE. A Treatise on the Theory of Bessel Functions, by G. N. Watson, Cambridge University Press, 1922, has been brought out while this volume is in press. This Treatise gives by far the most complete account of the theory and properties of Bessel Functions that exists, and should become the standard work on the subject with respect to notation. A particularly valuable feature is the Collection of Tables of Bessel Functions at the end of the volume and the Bibliography, giving references to all the important works on the subject.

**9.91** Tables of Legendre, Bessel and allied functions.

$P_n(x)$ **(9.001).**

[1] A second edition of Gray and Mathews' Treatise, prepared by A. Gray and T. M. MacRobert, has been published (1922) while this volume is in press. The notation of the first edition has been altered in some respects.

B. A. Report, 1879, pp. 54–57. Integral values of $n$ from 1 to 7; from $x = 0.01$ to $x = 1.00$, interval 0.01, 16 decimal places.

Jahnke and Emde: Funktionentafeln, p. 83; same to 4 decimal places.

$P_n(\cos\theta)$

Phil. Trans. Roy. Soc. London, 203, p. 100, 1904. Integral values of $n$ from 1 to 20, from $\theta = 0$ to $\theta = 90$, interval 5, 7 decimal places.

Phil. Mag. 32, p. 512, 1891. Integral values of $n$ from 1 to 7, $\theta = 0$ to $\theta = 90$, interval 1; 4 decimal places. Reproduced in Jahnke and Emde, p. 85.

Tallquist, Acta Soc. Sc. Fennicae, Helsingfors, 33, pp. 1–8. Integral values of $n$ from 1 to 8; $\theta = 0$ to $\theta = 90$, interval 1, 10 decimal places.

Airey, Proc. Roy. Soc. London, 96, p. 1, 1919. Tables by means of which zonal harmonics of high order may be calculated.

Lodge, Phil. Trans. Roy. Soc. London, 203, 1904, p. 87. Integral values of $n$ from 1 to 20; $\theta = 0$ to $\theta = 90$, interval 5, 7 decimal places. Reprinted in Rayleigh, Collected Works, Volume V, p. 162.

$$\frac{\partial P_n(\cos\theta)}{\partial\theta}.$$

Farr, Proc. Roy. Soc. London, 64, 199, 1899. Integral values of $n$ from 1 to 7; $\theta = 0$ to $\theta = 90$, interval 1, 4 decimal places. Reproduced in Jahnke and Emde, p. 88.

$J_0(x)$, $J_1(x)$ (**9.101**).

Meissel's tables, $x = 0.01$ to $x = 15.50$, interval 0.01, to 12 decimal places, are given in Table I of Gray and Mathews' Treatise on Bessel's Functions.

Aldis, Proc. Roy. Soc. London 66, 40, 1900. $x = 0.1$ to $x = 6.0$, interval 0.1, 21 decimal places.

Jahnke and Emde, Funktionentafeln, Table III. $x = 0.01$ to $x = 15.50$, interval 0.01, 4 decimal places.

$J_n(x)$ (**9.101**).

Gray and Mathews, Table II. Integral values of $n$ from $n = 0$ to $n = 60$; integral values of $x$ from $x = 1$ to $x = 24$, 18 decimal places.

Jahnke and Emde, Table XXIII, same, to 4 significant figures.

B. A. Report, 1915, p. 29; $n = 0$ to $n = 13$.

| | | |
|---|---|---|
| $x = 0.2$ to $x = 6.0$ | interval 0.2 | 6 decimal places, |
| $x = 6.0$ to $x = 16.0$ | interval 0.5 | 10 decimal places. |

Hague, Proc. London Physical Soc. 29, 211, 1916–17, gives graphs of $J_n(x)$ for integral values of $n$ from 0 to 12, and $n = 18$, $x$ ranging from 0 to 17.

$$-\frac{\pi}{2}Y_0(x) = G_0(x); \quad -\frac{\pi}{2}Y_1(x) = G_1(x).$$

B. A. Report, 1913, pp. 116–130. $x = 0.01$ to $x = 16.0$, interval 0.01, 7 decimal places.

B. A. Report, 1915, $x = 6.5$ to $x = 15.5$, interval 0.5, 10 decimal places.

Aldis, Proc. Roy. Soc. London, 66, 40, 1900: $x = 0.1$ to $x = 6.0$. Interval 0.1, 21 decimal places.

Jahnke and Emde, Tables VII and VIII, functions denoted $K_0(x)$ and $K_1(x)$, $x = 0.1$ to $x = 6.0$, interval 0.1; $x = 0.01$ to $x = 0.99$, interval 0.01; $x = 1.0$ to $x = 10.3$, interval 0.1; 4 decimal places.

$$-\frac{\pi}{2} Y_n(x) = G_n(x).$$

B. A. Report, 1914, p. 83. Integral values of $n$ from 0 to 13. $x = 0.01$ to $x = 6.0$, interval 0.1; $x = 6.0$ to $x = 16.0$, interval 0.5; 5 decimal places.

$$\frac{\pi}{2} Y_0(x) + (\log 2 - \gamma) J_0(x),$$

$$\frac{\pi}{2} Y_1(x) + (\log 2 - \gamma) J_1(x).$$

Denoted $Y_0(x)$ and $Y_1(x)$ respectively in the tables.

B. A. Report, 1914, p. 76, $x = 0.02$ to $x = 15.50$, interval 0.02, 6 decimal places.

B. A. Report, 1915, p. 33, $x = 0.1$ to $x = 6.0$, interval 0.1; $x = 6.0$ to $x = 15.5$, interval 0.5, 10 decimal places.

Jahnke and Emde, Table VI, $x = 0.01$ to $x = 1.00$, interval 0.01; $x = 1.0$ to $x = 10.2$, interval 0.1, 4 decimal places.

$Y_0(x)$, $Y_1(x)$. Denoted $N_0(x)$ and $N_1(x)$ respectively.

Jahnke and Emde, Table IX, $x = 0.1$ to $x = 10.2$, interval 0.1, 4 decimal places.

$$\frac{\pi}{2} Y_n(x) + (\log 2 - \gamma)\, J_n(x).$$

Denoted $Y_n(x)$ in tables.

B. A. Report, 1915. Integral values of $n$ from 1 to 13. $x = 0.2$ to $x = 6.0$, interval 0.2; $x = 6.0$ to $x = 15.5$, interval 0.5, 6 decimal places.

$J_{n+\frac{1}{2}}(x)$.

Jahnke and Emde, Table II. Integral values of $n$ from $n = 0$ to $n = 6$, and $n = -1$ to $n = -7$; $x = 0$ to $x = 50$, interval 1.0, 4 figures.

$J_{\frac{1}{3}}(x)$, $J_{-\frac{1}{3}}(x)$.

Watson, Proc. Roy. Soc. London, 94, 204, 1918.

$x = 0.05$ to $x = 2.00$ interval 0.05,
$x = 2.0$ to $x = 8.0$ interval 0.2,

4 decimal places.

$J_\alpha(\alpha)$, $J_{\alpha-1}(\alpha)$

$$-\frac{\pi}{2} Y_\alpha(\alpha), \quad -\frac{\pi}{2} Y_{\alpha-1}(\alpha).$$

Denoted $G_\alpha(\alpha)$ and $G_{\alpha-1}(\alpha)$ respectively.

$\frac{\pi}{2} Y_\alpha(\alpha) + (\log 2 - \gamma) J_\alpha(\alpha)$,

$\frac{\pi}{2} Y_{\alpha-1}(\alpha) + (\log 2 - \gamma) J_{\alpha-1}(\alpha)$. Denoted $-Y_\alpha(\alpha)$ and $-Y_{\alpha-1}(\alpha)$.

Tables of these six functions are given in the B. A. Report, 1916, as follows:

| From $\alpha$ | to $\alpha$ | Interval |
|---|---|---|
| 1 | 50 | 1 |
| 50 | 100 | 5 |
| 100 | 200 | 10 |
| 200 | 400 | 20 |
| 400 | 1000 | 50 |
| 1000 | 2000 | 100 |
| 2000 | 5000 | 500 |
| 5000 | 20000 | 1000 |
| 20000 | 30000 | 10000 |
| 100,000 | | |
| 500,000 | | |
| 1,000,000 | | |

$I_0(x)$, $I_1(x)$ (**9.211**).

Aldis, Proc. Roy. Soc. London, 64, pp. 218–223, 1899; $x = 0.1$ to $x = 6.0$, interval 0.1; $x = 6.0$ to $x = 11.0$, interval 1.0, 21 decimal places.

Jahnke and Emde, Tables XI and XII, 4 places:

$x = 0.01$ to $x = 5.10$ interval 0.01,
$x = 5.10$ to $x = 6.0$ interval 0.1,
$x = 6.0$ to $x = 11.0$ interval 1.0.

$\mathrm{I}_0(x)$ (**9.211**).

B. A. Report, 1896; $x = 0.001$ to $x = 5.100$, interval 0.001, 9 decimal places.

$\mathrm{I}_1(x)$ (**9.211**).

B. A. Report, 1893; $x = 0.001$ to $x = 5.100$, interval 0.001, 9 decimal places.

Gray and Mathews, Table V, $x = 0.01$ to $x = 5.10$, interval 0.01, 9 decimal places.

$\mathrm{I}_n(x)$ (**9.211**).

B. A. Report, 1889, pp. 28–32; integral values of $n$ from 0 to 11, $x = 0.2$ to $x = 6.0$, interval $0 > 2$, 12 decimal places. These tables are reproduced in Gray and Mathews, Table VI.

Jahnke and Emde, Table XXIV; same ranges, to 4 places.

$J_0(x\sqrt{i}) = X - iY$,
$\sqrt{2} J_1(x\sqrt{i}) = X_1 + iY_1$

Aldis, Proc. Roy. Soc. London, 66, 142, 1900; $x = 0.1$ to $x = 6.0$, interval 0.1, 21 decimal places.

Jahnke and Emde, Tables XV and XVI, same range, to 4 places.

$J_0(x\sqrt{i})$.

Gray and Mathews, Table IV; $x = 0.2$ to $x = 6.0$, interval 0.2, 9 decimal places.

$Y_0(x\sqrt{i})$ **(9.104)** Denoted $N_0(x\sqrt{i})$ in table.

$H_0^{\mathrm{I}}(x\sqrt{i})$, $H_1^{\mathrm{I}}(x\sqrt{i})$.

Jahnke and Emde, Tables XVII and XVIII; $x = 0.2$ to $x = 6.0$, interval 0.2, 4–7 figures.

$$\frac{i\pi}{2} H_0^{\mathrm{I}}(ix) = K_0(x),$$
$$-\frac{\pi}{2} H_0^{\mathrm{I}}(ix) = K_1(x), \qquad \textbf{(9.212)}.$$

Aldis, Proc. Roy. Soc. London, 64, 219–223, 1899; $x = 0.1$ to $x = 12.0$, interval 0.1, 21 decimal places.

Jahnke and Emde, Table XIV; same, to 4 places.

$iH_0^{\mathrm{I}}(ix)$, $-H_0^{\mathrm{I}}(ix)$ **(9.107)**.

Jahnke and Emde, Table XIII; $x = 0.12$ to $x = 6.0$, interval 0.2, 4 figures.

$\operatorname{ber} x$, $\operatorname{ber}' x$,
$\operatorname{bei} x$, $\operatorname{bei}' x$, **(9.215)**.

B. A. Report, 1912; $x = 0.1$ to $x = 10.0$, interval 0.1, 9 decimal places.

Jahnke and Emde, Table XX; $x = 0.5$ to $x = 6.0$, interval 0.5, and $x = 8$, 10, 15, 20, 4 decimal places.

$\operatorname{ker} x$, $\operatorname{ker}' x$,
$\operatorname{kei} x$, $\operatorname{kei}' x$, **(9.216)**.

B. A. Report, 1915; $x = 0.1$ to $x = 10.0$, interval 0.1, 7–10 decimal places.

$\operatorname{ber}^2 x + \operatorname{bei}^2 x$,
$\operatorname{ber}'^2 x + \operatorname{bei}'^2 x$,
$\operatorname{ber} x \operatorname{bei}' x - \operatorname{bei} x \operatorname{ber}' x$, and the corresponding ker and kei
$\operatorname{ber} x \operatorname{ber}' x + \operatorname{bei} x \operatorname{bei}' x$, functions.

B. A. Report, 1916; $x = 0.2$ to $x = 10.0$, interval 0.2, decimal places.

$S_n(x)$, $S'_n(x)$, $\log S_n(x)$, $\log S'_n(x)$,
$C_n(x)$, $C'_n(x)$, $\log C_n(x)$, $\log C'_n(x)$, **(9.261)**.
$E_n(x)$, $E'_n(x)$, $\log E_n(x)$, $\log E'_n(x)$,

B. A. Report, 1916; integral values of $n$ from 0 to 10, $x = 1.1$ to $x = 1.9$, interval 0.1, 7 decimal places.

$$G(x) = -\sqrt{2}\,\Pi\left(\frac{1}{4}\right) x^{-\frac{1}{4}} J_{\frac{1}{4}}\left(\frac{x}{2}\right) = -\frac{1}{0.78012}\, x^{-\frac{1}{4}} J_{\frac{1}{4}}\left(\frac{x}{2}\right)$$

$$D(x) = \frac{1}{\sqrt{2}}\,\Pi\left(-\frac{1}{4}\right) x^{\frac{1}{4}} J_{-\frac{1}{4}}\left(\frac{x}{2}\right) = \frac{1}{1.15407}\, x^{\frac{1}{4}} J_{-\frac{1}{4}}\left(\frac{x}{2}\right)$$

Table I of Jahnke and Emde gives these two functions to 3 decimal places for $x = 0.2$ to $x = 8.0$, interval 0.2, and $x = 8.0$ to $x = 12.0$, interval 1.0.

Roots of $J_0(x) = 0$.

Airey, Phil. Mag. 36, p. 241, 1918: First 40 roots ($\rho$) with corresponding values of $J_1(\rho)$, 7 decimal places.

Jahnke and Emde, Table IV, same, to 4 decimal places.

Roots of $J_1(x) = 0$.

Gray and Mathews, Table III, first 50 roots, with corresponding values of $J_0(x)$, 16 decimal places.

Airey, Phil. Mag. 36, p. 241: First 40 roots ($r$) with corresponding values of $J_0(r)$, 7 decimal places.

Jahnke and Emde, Table IV, same, to 4 decimal places.

Roots of $J_n(x) = 0$.

B. A. Report, 1917, first 10 roots, to 6 figures, for the following integral values of $n$: 0–10, 15, 20, 30, 40, 50, 75, 100, 200, 300, 400, 500, 750, 1000.

Jahnke and Emde, Table XXII, first 9 roots, 3 decimal places, integral values of $n$ 0–9.

Roots of:

$(\log 2 - \gamma)J_n(x) + \frac{\pi}{2} Y_n(x) = 0.$ Denoted $Y_n(x) = 0$ in table.

Airey: Proc. London Phys. Soc. 23, p. 219, 1910–11. First 40 roots for $n = 0, 1, 2, 5$ decimal places.

Jahnke and Emde, Table X, first 4 roots for $n = 0, 1$. $E$ decimal places.

Roots of:

$Y_0(x) = 0$,
$Y_1(x) = 0$. Denoted $N_0(x)$ and $N_1(x)$ in tables.

Airey: l. c. First 10 roots, 5 decimal places.

Roots of:

$J_0(x) \pm (\log 2 - \gamma)J_0(x) + \frac{\pi}{2} Y_0(x) = 0.$ Denoted $J_0(x) \pm Y_0(x) = 0.$

$J_1(x) + (\log 2 - \gamma)J_1(x) + \frac{\pi}{2} Y_1(x) = 0.$ Denoted $J_1(x) + Y_1(x) = 0.$

$J_0(x) - 2(\log 2 - \gamma)J_0(x) + \frac{\pi}{2} Y_0(x) = 0.$ Denoted $J_0(x) - 2Y_0(x) = 0.$

$10J_0(x) \pm (\log 2 - \gamma)J_0(x) + \frac{\pi}{2} Y_0(x) = 0.$ Denoted $10J_0(x) \pm Y_0(x) = 0.$

Airey, l. c. First 10 roots, 5 decimal places.

Roots of

$$\frac{J_{n+1}(x)}{J_n(x)} + \frac{I_{n+1}(x)}{I_n(x)} = 0.$$

Airey, l. c. First 10 roots: $n = 0$, 4 decimal places, $n = 1, 2, 3$, 3 decimal places.

Jahnke and Emde, Table XXV, first 5 roots for $n = 0$, 3 for $n = 1$, 2 for $n = 2$: 4 figures.

Airey, l. c. gives roots of some other equations involving Bessel's functions connected with the vibration of circular plates.

Roots of:

$$J_\nu(x)Y_\nu(x) = J_\nu(kx)Y_\nu(kx).$$

Jahnke and Emde, Table XXVI, first 6 roots, 4 decimal places, for $\nu = 0, 1/2, 1, 3/2, 2, 5/2$: $k = 1.2, 1.5, 2.0$.

Table XXVIII, first root, multiplied by $(k - 1)$ for $k = 1, 1.2, 1.5, 2-11, 19, 39, \infty$: $\nu$ same as above.

Table XXIX, first 4 roots, multiplied by $(k - 1)$ for certain irrational values of $k$, and $\nu = 0, 1$.

# X. NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

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## INTRODUCTION

Differential equations are usually first encountered in the final chapter of a book on integral calculus. The methods which are there given for solving them are essentially the same as those employed in the calculus. Similar methods are used in the first special work on the subject. That is, numerous types of differential equations are given in which the variables can be separated by suitable devices; little or nothing is said about the existence of solutions of other types, or about methods of finding the solutions. The false impression is often left that only exceptionally can differential equations be solved. Whatever satisfaction there may be in learning that some problems in geometry and physics lead to standard forms of differential equations is more than counterbalanced by the discovery that most practical problems do not lead to such forms.

**10.01** The point of view adopted here and the methods which are developed can be best understood by considering first some simpler and better known mathematical theories. Suppose

1. $$F(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

is a polynomial equation in $x$ having real coefficients $a_1, a_2, \ldots, a_n$. If $n$ is 1, 2, 3, or 4 the values of $x$ which satisfy the equation can be expressed as explicit functions of the coefficients. If $n$ is greater than 4, formulas for the solution can not in general be written down. Nevertheless, it is possible to prove that $n$ solutions exist and that at least one of them is real if $n$ is odd. If the coefficients are given numbers, there are straightforward, though somewhat laborious, methods of finding the solutions. That is, even though general formulas for the solutions are not known, yet it is possible both to prove the existence of the solutions and also to find them in any special numerical case.

**10.02** Consider as another illustration the definite integral

1. $$I = \int_a^b f(x)\,dx,$$

where $f(x)$ is continuous for $a \leqslant x \leqslant b$. If $F(x)$ is such a function that

2. $$\frac{dF}{dx} = f(x),$$

then $I = F(b) - F(a)$. But suppose no $F(x)$ can be found satisfying (2). It is nevertheless possible to prove that the integral $I$ exists, and if the value of $(x)$ is given for every value of $x$ in the interval $a \leqslant x \leqslant b$, it is possible to find the numerical value of $I$ with any desired degree of approximation. That is, it is not necessary that the primitive of the integrand of a definite integral be known in order to prove the existence of the integral, or even to find its value in any particular example.

**10.03** The facts are analogous in the case of differential equations. Those having numerical coefficients and prescribed initial conditions can be solved regardless of whether or not their variables can be separated. They need to satisfy only mild conditions which are always fulfilled in physical problems. It is with a sense of relief that one finds he can solve, numerically, any particular problem which can be expressed in terms of differential equations.

**10.04** This chapter will contain an account of a method of solving ordinary differential equations which is applicable to a broad class including all those which arise in physical problems. A large amount of experience has shown that the method is very convenient in practice. It must be understood that there is for it an underlying logical basis, involving refinements of modern analysis, which fully justifies the procedure. In other words, it can be proved that the process is capable of furnishing the solution with any desired degree of accuracy. The proofs of these facts belong to the domain of pure analysis and will not be given here.

**10.10** Simpson's Method of Computing Definite Integrals. The method of solving differential equations which will be given later involves the computation of definite integrals by a special process which will be developed in this and the following sections.

Let $t$ be the variable of integration, and consider the definite integral

$$1. \qquad F = \int_a^b f(t)\, dt.$$

This integral can be interpreted as the area between the $t$-axis and the curve $y = f(t)$ and bounded by the ordinates $t = a$ and $t = b$, figure 1.

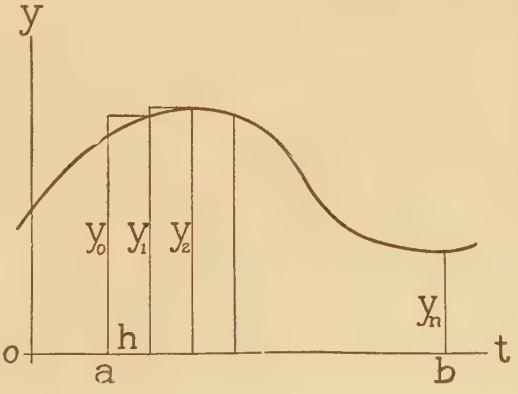

FIG. 1

Let $t_0 = a$, $t_n = b$, $y_i = f(t_i)$, and divide the interval $a \leqslant t \leqslant b$ up into $n$ equal parts, each of length $h = (b - a)/n$. Then an approximate value of $F$ is

$$2. \qquad F_0 = h(y_1 + y_2 + \cdots + y_n).$$

This is the sum of rectangles whose ordinates, figure 1, are $y_1, y_2, \ldots, y_n$.

**10.11** A more nearly exact value can be obtained for the first two intervals, for example, by putting a curve of the second degree through the three points

$y_0$, $y_1$, $y_2$, and finding the area between the $t$-axis and this curve and bounded by the ordinates $t_0$ and $t_2$. The equation of the curve is

1. $$y = a_0 + a_1(t - t_0) + a_2(t - t_0)^2,$$

where the coefficients $a_0$, $a_1$, and $a_2$ are determined by the conditions that $y$ shall equal $y_0$, $y_1$, and $y_2$ at $t$ equal to $t_0$, $t_1$ and $t_2$ respectively; or

2. $$\begin{cases} y_0 = a_0, \\ y_1 = a_0 + a_1(t_1 - t_0) + a_2(t_1 - t_0)^2, \\ y_2 = a_0 + a_1(t_2 - t_0) + a_2(t_2 - t_0)^2. \end{cases}$$

It follows from these equations and $t_2 - t_1 = t_1 - t_0 = h$ that

3. $$\begin{cases} a_0 = y_0, \\ a_1 = -\frac{1}{2h}(3y_0 - 4y_1 + y_2), \\ a_2 = \frac{1}{2h^2}(y_0 - 2y_1 + y_2). \end{cases}$$

The definite integral $\int_{t_0}^{t_2} y\,dt$ is approximately

$$I = \int_{t_0}^{t_2}\left[a_0 + a_1(t - t_0) + a_2(t - t_0)^2\right] dt = 2h\left[a_0 + a_1 h + \frac{4}{3}a_2 h^2\right],$$

which becomes as a consequence of (3)

4. $$I = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

**10.12** The value of the integral over the next two intervals, or from $t_2$ to $t_4$, can be computed in the same way. If $n$ is even, the approximate value of the integral from $t_0$ to $t_n$ is therefore

$$F_1 = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots\cdots + 4y_{n-1} + y_n].$$

This formula, which is due to Simpson, gives results which are usually remarkably accurate considering the simplicity of the arithmetical operations.

**10.13** If a curve of the third degree had been passed through the four points $y_0$, $y_1$, $y_2$, and $y_3$, the integral corresponding to (4), but over the first three intervals, would have been found to be

$$I = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3].$$

**10.20** Digression on Difference Functions. For later work it will be necessary to have some properties of the successive differences of the values of a function for equally spaced values of its argument.

As before, let $y_i$ be the value of $f(t)$ for $t = t_i$. Then let

$$\Delta_1 y_1 = y_1 - y_0,$$
$$\Delta_1 y_2 = y_2 - y_1,$$
$$\cdots\cdots\cdots$$
$$\Delta_1 y_n = y_n - y_{n-1},$$
$$\cdots\cdots\cdots\cdots$$

These are the first differences of the values of the function $y$ for successive values of $t$. All the successive intervals for $t$ are supposed to be equal.

**10.21** In a similar way the second differences are defined by

$$\Delta_2 y_2 = \Delta_1 y_2 - \Delta_1 y_1,$$
$$\Delta_2 y_3 = \Delta_1 y_3 - \Delta_1 y_2,$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta_2 y_n = \Delta_1 y_n - \Delta_1 y_{n-1},$$
$$\cdots\cdots\cdots\cdots$$

**10.22** In a similar way third differences are defined by

$$\Delta_3 y_3 = \Delta_2 y_3 - \Delta_2 y_2,$$
$$\Delta_3 y_4 = \Delta_2 y_4 - \Delta_2 y_3,$$
$$\cdots\cdots\cdots$$
$$\Delta_3 y_n = \Delta_2 y_n - \Delta_2 y_{n-1},$$
$$\cdots\cdots\cdots\cdots,$$

and obviously the process can be repeated as many times as may be desired.

**10.23** The table of successive differences can be formed conveniently from the tabular values of the function and can be arranged in a table as follows:

TABLE I

| $y$ | $\Delta_1 y$ | $\Delta_2 y$ | $\Delta_3 y$ |
|---|---|---|---|
| $y_0$ | | | |
| $y_1$ | $\Delta_1 y_1$ | | |
| $y_2$ | $\Delta_1 y_2$ | $\Delta_2 y_2$ | |
| $y_3$ | $\Delta_1 y_3$ | $\Delta_2 y_3$ | $\Delta_3 y_3$ |
| ..... | ..... | ..... | ..... |

In this table the numbers in each column are subtracted from those immediately below them and the remainders are placed in the next column to the right on the same line as the minuends. Variations from this precise arrangement could be, and indeed often have been, adopted.

**10.24** A very important advantage of a table of differences is that it is almost sure to reveal any errors that may have been committed in computing the $y_i$. If a single $y_i$ has an error $\epsilon$, it follows from **10.20** that the first difference $\Delta_1 y_i$ will contain the error $+\epsilon$ and $\Delta_1 y_{i+1}$ will contain the error $-\epsilon$. But the second differences $\Delta_2 y_i$, $\Delta_2 y_{i+1}$, and $\Delta_2 y_{i+2}$ will contain the respective errors $+\epsilon$, $-2\epsilon$, $+\epsilon$. Similarly, the third differences $\Delta_3 y_i$, $\Delta_3 y_{i+1}$, $\Delta_3 y_{i+2}$, and $\Delta_3 y_{i+3}$ will contain the respective errors $+\epsilon$, $-3\epsilon$, $+3\epsilon$, $-\epsilon$. An error in a single $y_i$ affects $j+1$ differences of order $j$, and the coefficients of the error are the binomial coefficients with alternating signs. The algebraic sums of the errors in the affected

numbers in the various difference columns are zero. Now in such functions as ordinarily occur in practice the numerical values of the differences, if the intervals are not too great, decrease with rapidity and run smoothly. If an error is present, however, the differences of higher order become very irregular.

**10.25** As an illustration, consider the function $y = \sin t$ for $t$ equal to 10°, 15°, . . . . . The following table gives the function and its successive differences, expressed in terms of units of the fourth decimal:[1]

TABLE II

| $t$ | $\sin t$ | $\Delta_1 \sin t$ | $\Delta_2 \sin t$ | $\Delta_3 \sin t$ |
|---|---|---|---|---|
| 10° | 1736 | | | |
| 15 | 2588 | 852 | | |
| 20 | 3420 | 832 | −20 | |
| 25 | 4226 | 806 | −26 | −6 |
| 30 | 5000 | 774 | −32 | −6 |
| 35 | 5736 | 736 | −38 | −6 |
| 40 | 6428 | 692 | −44 | −6 |
| 45 | 7071 | 643 | −49 | −5 |
| 50 | 7660 | 589 | −54 | −5 |
| 55 | 8191 | 531 | −58 | −4 |
| 60 | 8660 | 469 | −62 | −4 |
| 65 | 9063 | 403 | −66 | −4 |
| 70 | 9397 | 334 | −69 | −3 |

Suppose, however, that an error of two units had been made in determining the sine of 45° and that 7073 had been taken in place of 7071. Then the part of the table adjacent to this number would have been the following:

TABLE III

| $t$ | $\sin t$ | $\Delta_1 \sin$ | $\Delta_2 \sin t$ | $\Delta_3 \sin t$ |
|---|---|---|---|---|
| 25° | 4226 | | | |
| 30 | 5000 | 774 | | |
| 35 | 5736 | 736 | −38 | |
| 40 | 6428 | 692 | −44 | − 6 |
| 45 | 7073 | 645 | −47 | − 3 |
| 50 | 7660 | 587 | −58 | −11 |
| 55 | 8191 | 531 | −56 | + 2 |
| 60 | 8660 | 469 | −62 | − 6 |
| 65 | 9063 | 403 | −66 | − 4 |

The irregularity in the numbers of the last column shows the existence of an error, and, in fact, indicates its location. In the third differences four numbers

[1] Often it is not necessary to carry along the decimal and zeros to the left of the first significant figure.

will be affected by an error in the value of the function. The erroneous numbers in the last column are clearly the second, third, fourth, and fifth. The algebraic sum of these four numbers equals the sum of the four correct numbers, or $-18$. Their average is $-4.5$. Hence the central numbers are probably $-5$ and $-4$. Since the errors in these numbers are $-3\epsilon$ and $+3\epsilon$, it follows that $\epsilon$ is probably $+2$. The errors in the second and fifth numbers are $+\epsilon$ and $-\epsilon$ respectively. On making these corrections and working back to the first column, it is found that 7073 should be replaced by 7071.

**10.30** Computation of Definite Integrals by Use of Difference Functions.

Suppose the values of $f(t)$ are known for $t = t_{n-2}$, $t_{n-1}$, $t_n$, and $t_{n+1}$. Suppose it is desired to find the integral

$$I_n = \int_{t_n}^{t_{n+1}} f(t)\,dt. \tag{1}$$

The coefficients $b_0$, $b_1$, $b_2$, and $b_3$ of the polynomial can be determined, as above, so that the function

$$y = b_0 + b_1(t - t_n) + b_2(t - t_n)^2 + b_3(t - t_n)^3 \tag{2}$$

shall take the same values as $f(t)$ for $t = t_{n-2}$, $t_{n-1}$, $t_n$, and $t_{n+1}$.

With this approximation to the function $f(t)$, the integral becomes (since $t_{n+1} - t_n = h$)

$$I_n = \int_{t_n}^{t_{n+1}} [b_0 + b_1(t - t_n) + b_2(t - t_n)^2 + b_3(t - t_n)^3]\,dt \tag{3}$$

$$= h[b_0 + \tfrac{1}{2} b_1 h + \tfrac{1}{3} b_2 h^2 + \tfrac{1}{4} b_3 h^3].$$

The coefficients $b_0$, $b_1$, $b_2$, and $b_3$ will now be expressed in terms of $y_{n+1}$, $\Delta_1 y_{n+1}$, $\Delta_2 y_{n+1}$, and $\Delta_3 y_{n+1}$. It follows from (2) that

$$\begin{cases} y_{n-2} = b_0 - 2b_1 h + 4b_2 h^2 - 8b_3 h^3, \\ y_{n-1} = b_0 - b_1 h + b_2 h^2 - b_3 h^3, \\ y_n = b_0, \\ y_{n+1} = b_0 + b_1 h + b_2 h^2 + b_3 h^3. \end{cases} \tag{4}$$

Then it follows from the rules for determining the difference functions that

$$\begin{cases} \Delta_1 y_{n-1} = b_1 h - 3b_2 h^2 + 7b_3 h^3, \\ \Delta_1 y_n = b_1 h - b_2 h^2 + b_3 h^3, \\ \Delta_1 y_{n+1} = b_1 h + b_2 h^2 + b_3 h^3. \end{cases} \tag{5}$$

$$\begin{cases} \Delta_2 y_n = 2b_2 h^2 - 6b_3 h^3, \\ \Delta_2 y_{n+1} = 2b_2 h^2. \end{cases} \tag{6}$$

$$\Delta_3 y_{n+1} = 6b_3 h^3. \tag{7}$$

It follows from the last equations of these four sets of equations that

$$8.\qquad \begin{cases} b_0 = y_{n+1} - \Delta_1 y_{n+1}, \\ b_1 h = \Delta_1 y_{n+1} - \frac{1}{2}\Delta_2 y_{n+1} - \frac{1}{6}\Delta_3 y_{n+1}, \\ b_2 h^2 = \frac{1}{2}\Delta_2 y_{n+1}, \\ b_3 h^3 = \frac{1}{6}\Delta_3 y_{n+1}. \end{cases}$$

Therefore the integral (3) becomes

$$9.\qquad I_n = h\left[y_{n+1} - \tfrac{1}{2}\Delta_1 y_{n+1} - \tfrac{1}{12}\Delta_2 y_{n+1} - \tfrac{1}{24}\Delta_3 y_{n+1} - \ldots\right].$$

The coefficients of the higher order terms $\Delta_4 y_{n+1}$ and $\Delta_5 y_{n+1}$ are $-\frac{19}{720}$ and $\frac{1}{48}$ respectively.

**10.31** Obviously, if it were desired, the integral from $t_{n-2}$ to $t_{n-1}$, or over any other part of this interval, could be computed by the same methods. For example, the integral from $t_{n-1}$ to $t_n$ is

$$I_{n-1} = \int_{t_{n-1}}^{t_n} f(t)\,dt,$$
$$= h\left[y_{n+1} - \tfrac{3}{2}\Delta_1 y_{n+1} + \tfrac{5}{12}\Delta_2 y_{n+1} + \tfrac{1}{24}\Delta_3 y_{n+1} + \ldots\right].$$

## NUMERICAL ILLUSTRATIONS

**10.32** Consider first the application of Simpson's method. Suppose it is required to find

$$I = \int_{25^\circ}^{55^\circ} \sin t\,dt = -\left[\cos t\right]_{25^\circ}^{55^\circ} = 0.3327.$$

On applying **10.12** with the numbers taken from Table I, it is found that

$$I_1 = \frac{5^\circ}{3}[.4226 + 2.0000 + 1.1472 + 2.5712 + 1.4142 + 3.0640 + .8191],$$

which becomes, on reducing 5° to radians,

$$I_1 = 0.3327,$$

agreeing to four places with the correct result.

**10.33** On applying **10.11** (4) and omitting alternate entries in Table II, it is found that

$$I = \int_{25^\circ}^{45^\circ} \sin t\,dt = \frac{10^\circ}{3}[.4226 + 2.2944 + .7071] = 0.1992,$$

which is also correct to four places. These formulas could hardly be surpassed in ease and convenience of application.

**10.34** Now consider the application of **10.30** (9). As it stands it furnishes the integral over the single interval $t_n$ to $t_{n+1}$. If it is desired to find the integral from $t_n$ to $t_{n+m}$, the formula for doing so is obviously the sum of $m$ formulas such as (9), the value of the subscript going from $n+1$ to $n+m+1$, or

$$I_{n,m} = h\left[\left(y_{n+1} + \cdots + y_{n+m+1}\right) - \frac{1}{2}\left(\Delta_1 y_{n+1} + \cdots + \Delta_1 y_{n+m+1}\right)\right.$$

$$\left. - \frac{1}{12}\left(\Delta_2 y_{n+1} + \cdots + \Delta_2 y_{n+m+1}\right) - \frac{1}{24}\left(\Delta_3 y_{n+1} + \cdots + \Delta_3 y_{n+m+1}\right) + \cdots\right].$$

On applying this formula to the numbers of Table I, it is found that

$$\begin{aligned} I = \int_{25^\circ}^{55^\circ} \sin t\, dt = 5^\circ[&(.5000 + .5736 + .6428 + .7071 + .7660 + .8191) \\ &- \frac{1}{2}(.0774 + .0736 + .0692 + .0643 + .0589 + .0531) \\ &+ \frac{1}{12}(.0032 + .0038 + .0044 + .0049 + .0054 + .0058) \\ &+ \frac{1}{24}(.0006 + .0006 + .0006 + .0005 + .0005 + .0004)] \\ &= 0.3327, \end{aligned}$$

agreeing to four places with the exact value. When a table of differences is at hand covering the desired range this method involves the simplest numerical operations. It must be noted, however, that some of the required differences necessitate a knowledge of the value of the function for earlier values of the argument than the lower limit of the integral.

**10.40** Reduced Form of the Differential Equations. Differential equations which arise from physical problems usually involve second derivatives. For example, the differential equation satisfied by the motion of a vibrating tuning fork has the form

$$\frac{d^2x}{dt^2} = -kx,$$

where $k$ is a constant depending on the tuning fork.

**10.41** The differential equations for the motion of a body subject to gravity and a retardation which is proportional to its velocity are

$$\begin{cases} \dfrac{d^2x}{dt^2} = -c\dfrac{dx}{dt}, \\ \dfrac{d^2y}{dt^2} = -c\dfrac{dy}{dt} - g, \end{cases}$$

where $c$ is a constant depending on the resisting medium and the mass and shape of the body, while $g$ is the acceleration of gravity.

**10.42** The differential equations for the motion of a body moving subject to the law of gravitation are

$$\begin{cases} \dfrac{d^2x}{dt^2} = -k^2\dfrac{x}{r^3}, \\ \dfrac{d^2y}{dt^2} = -k^2\dfrac{y}{r^3}, \\ \dfrac{d^2z}{dt^2} = -k^2\dfrac{z}{r^3}, \\ r^2 = x^2 + y^2 + z^2. \end{cases}$$

**10.43** These examples illustrate sufficiently the types of differential equations which arise in practical problems. The number of the equations depends on the problem and may be small or great. In the problem of three bodies there are nine equations. The equations are usually not independent as is illustrated in **10.42,** where each equation involves all three variables $x$, $y$, and $z$ through $r$. On the other hand, equations **10.41** are mutually independent for the first does not involve $y$ or its derivatives and the second does not involve $x$ or its derivatives. The right members may involve $x$, $y$, and $z$ as is the case in **10.42,** or they may involve the first derivatives, as is the case in **10.41,** or they may involve both the coördinates and their first derivatives. In some problems they also involve the independent variable $t$.

**10.44** Hence physical problems usually lead to differential equations which are included in the form

$$\begin{cases} \dfrac{d^2x}{dt^2} = f\left(x, y, \dfrac{dx}{dt}, \dfrac{dy}{dt}, t\right), \\ \dfrac{d^2y}{dt^2} = g\left(x, y, \dfrac{dx}{dt}, \dfrac{dy}{dt}, t\right), \end{cases}$$

where $f$ and $g$ are functions of the indicated arguments. Of course, the number of equations may be greater than two.

**10.45** If we let

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt},$$

equations **10.44** can be written in the form

$$\begin{cases} \dfrac{dx}{dt} = x', \\ \dfrac{dx'}{dt} = f(x, y, x', y', t), \\ \dfrac{dy}{dt} = y', \\ \dfrac{dy'}{dt} = g(x, y, x', y', t). \end{cases}$$

**10.46** If we let $x = x_1$, $x' = x_2$, $y = x_3$, $y' = x_4$, . . . . . equations **10.45** are included in the form

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n, t), \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n, t). \end{cases}$$

This is the final standard form to which it will be supposed the differential equations are reduced.

**10.50** Definition of a Solution of Differential Equations. For simplicity in writing, suppose the differential equations are two in number and write them in the form

$$1. \qquad \begin{cases} \dfrac{dx}{dt} = f(x, y, t), \\ \dfrac{dy}{dt} = g(x, y, t), \end{cases}$$

where $f$ and $g$ are known functions of their arguments. Suppose $x = a$, $y = b$ at $t = 0$. Then

$$2. \qquad \begin{cases} x = \phi(t), \\ y = \psi(t), \end{cases}$$

is the solution of (1) satisfying these initial conditions if $\phi$ and $\psi$ are such functions that

$$3. \qquad \begin{aligned} \phi(0) &= a, \\ \psi(0) &= b, \\ \frac{d\phi}{dt} &= f(\phi, \psi, t), \\ \frac{d\psi}{dt} &= g(\phi, \psi, t), \end{aligned}$$

the last two equations being satisfied for all $0 \leqslant t \leqslant T$, where $T$ is a positive constant, the largest value of $t$ for which the solution is determined. It is not necessary that $\phi$ and $\psi$ be given by any formulas — it is sufficient that they have the properties defined by (3). Solutions *always exist*, though it will not be proved here, *if f and g are continuous functions of t and have derivatives with respect to both x and y*.

**10.51** Geometrical Interpretation of a Solution of Differential Equations. Geometrical interpretations of definite integrals have been of great value not only in leading to an understanding of their real meaning but also in suggesting

practical means of obtaining their numerical values. The same things are true in the case of differential equations.

For simplicity in the geometrical representation, consider a single equation

1. $$\frac{dx}{dt} = f(x, t),$$

where $x = a$ at $t = 0$. Suppose the solution is

2. $$x = \phi(t),$$

Equation (2) defines a curve whose coördinates are $x$ and $t$. Suppose it is represented by figure 2. The value of the tangent to the curve at every point on it is given by equation (1), for there is, corresponding to each point, a pair of values of $x$ and $t$ which gives $\frac{dx}{dt}$, the value of the tangent, when substituted in the right member of equation (1).

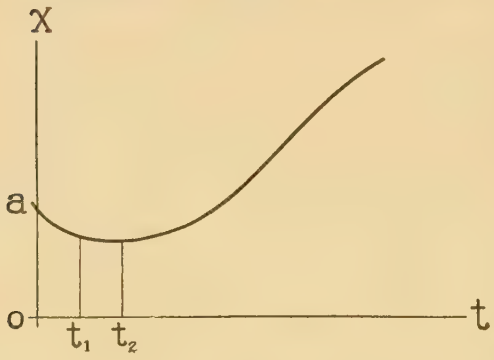


FIG. 2

Consider the initial point on the curve, viz. $x = a$, $t = 0$. The tangent at this point is $f(a, 0)$. The curve lies close to the tangent for a short distance from the initial point. Hence an approximate value of $x$ at $t = t_1$, $t_1$ being small, is the ordinate of the point where the tangent at $a$ intersects the line $t = t_1$, or

$$x_1 = f(a, 0)t_1.$$

The tangent at $x_1$, $t_1$ is defined by (1), and a new step in the solution can be made in the same way. Obviously the process can be continued as long as $x$ and $t$ have values for which the right member of (1) is defined. And the same process can be applied when there are any number of equations. While the steps of this process can be taken so short that it will give the solution with any desired degree of accuracy, it is not the most convenient process that may be employed. It is the one, however, which makes clearest to the intuitions the nature of the solution.

**10.6** Outline of the Method of Solution. Consider equations **10.50** (1) and their solution (2). The problem is to find functions $\phi$ and $\psi$ having the properties (2). If we integrate the last two equations of **10.50** (3) we shall have

1. $$\begin{cases} \phi = a + \int_0^t f(\phi, \psi, t)\, dt, \\ \psi = b + \int_0^t g(\phi, \psi, t)\, dt. \end{cases}$$

The difficulty arises from the fact that $\phi$ and $\psi$ are not known in advance and the integrals on the right can not be formed. Since $\phi$ and $\psi$ are the solution values of $x$ and $y$, we may replace them by the latter in order to preserve the original notation, and we have

$$2. \qquad \begin{cases} x = a + \int_0^t f(x, y, t)\, dt, \\ y = b + \int_0^t g(x, y, t)\, dt. \end{cases}$$

If $x$ and $y$ do not change rapidly in numerical value, then $f(x, y, t)$ and $g(x, y, t)$ will not in general change rapidly, and a first approximation to the values of $x$ and $y$ satisfying equations (2) is

$$3. \qquad \begin{cases} x_1 = a + \int_0^t f(a, b, t)\, dt, \\ y_1 = b + \int_0^t g(a, b, t)\, dt, \end{cases}$$

at least for values of $t$ near zero. Since $a$ and $b$ are constants, the integrands in (3) are known and the integrals can be computed. If the primitives can not be found the integrals can be computed by the methods of **10.1** or **10.3.**

After a first approximation has been found a second approximation is given by

$$4. \qquad \begin{cases} x_2 = a + \int_0^t f(x_1, y_1, t)\, dt, \\ y_2 = b + \int_0^t g(x_1, y_1, t)\, dt. \end{cases}$$

The integrands are again known functions of $t$ because $x_1$ and $y_1$ were determined as functions of $t$ by equations (3). Consequently $x_2$ and $y_2$ can be computed. The process can evidently be repeated as many times as is desired. The $n$th approximation is

$$5. \qquad \begin{cases} x_n = a + \int_0^t f(x_{n-1}, y_{n-1}, t)\, dt, \\ y_n = b + \int_0^t g(x_{n-1}, y_{n-1}, t)\, dt. \end{cases}$$

There is no difficulty in carrying out the process, but the question arises whether it converges to the solution. The answer, first established by Picard, is that, as $n$ increases, $x_n$ and $y_n$ tend toward the solution for all values of $t$ for which all the approximations belong to those values of $x$, $y$, and $t$ for which $f$ and $g$ have the properties of continuity with respect to $t$ and differentiability with respect to $x$ and $y$. If, for example, $f = \frac{\sin x}{x^2}$ and the value of $x_n$ tends towards zero for $t = T$, then the solution can not be extended beyond $t = T$.

It is found in practice that the longer the interval over which the integration is extended in the successive approximations, the greater the number of approximations which must be made in order to obtain a given degree of accuracy. In fact, it is preferable to take first a relatively short interval and to find the solution over this interval with the required accuracy, and then to continue from the end values of this interval over a new interval. This is what is done in actual work. The details of the most convenient methods of doing it will be explained in the succeeding sections.

**10.7** The Step-by-Step Construction of the Solution. Suppose the differential equations are

$$1 \qquad \begin{cases} \dfrac{dx}{dt} = f(x, y, t), \\ \dfrac{dy}{dt} = g(x, y, t), \end{cases}$$

with the initial conditions $x = a$, $y = b$ at $t = 0$. It is more difficult to start a solution than it is to continue one after the first few steps have been made. Therefore, it will be supposed in this section that the solution is well under way, and it will be shown how to continue it. Then the method of starting a solution will be explained in the next section, and the whole process will be illustrated numerically in the following one.

Suppose the values of $x$ and $y$ have been found for $t = t_1, t_2, \ldots, t_n$. Let them be respectively $x_1, y_1$; $x_2, y_2$; . . .; $x_n, y_n$, care being taken not to confuse the subscripts with those used in section **10.6** in a different sense. Suppose the intervals $t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}$ are all equal to $h$ and that it is desired to find the values of $x$ and $y$ at $t_{n+1}$, where $t_{n+1} - t_n = h$.

It follows from this notation and equations (2) of **10.6** that the desired quantities are

$$2. \qquad \begin{cases} x_{n+1} = x_n + \displaystyle\int_{t_n}^{t_{n+1}} f(x, y, t)\, dt, \\ y_{n+1} = y_n + \displaystyle\int_{t_n}^{t_{n+1}} g(x, y, t)\, dt. \end{cases}$$

The values of $x$ and $y$ in the integrands are of course unknown. They can be found by successive approximations, and if the interval is short, as is supposed, the necessary approximations will be few in number.

A fortunate circumstance makes it possible to reduce the number of approximations. The values of $x$ and $y$ are known at $t = t_n, t_{n-1}, t_{n-2}, \ldots$ From these values it is possible to determine in advance, by extrapolation, very close approximations to $x$ and $y$ for $t = t_{n+1}$. The corresponding values of $f$ and $g$ can be computed because these functions are given in terms of $x$, $y$, and $t$. They are also given for $t = t_n, t_{n-1}, \ldots$ Consequently, curves for $f$ and $g$ agreeing with their values at $t = t_{n+1}, t_n, t_{n-1}, \ldots$ can be constructed and the integrals (2) can be computed by the methods of **10.1** and **10.3**.

The method of extrapolating values of $x_{n+1}$ and $y_{n+1}$ must be given. Since the method is the same for both, consider only the former. Since, by hypothesis, $x$ is known for $t = t_n, t_{n-1}, t_{n-2}, \ldots$ the values of $x_n$, $\Delta_1 x_n$, $\Delta_2 x_n$, and $\Delta_3 x_n$ are known. If the interval $h$ is not too large the value of $\Delta_3 x_{n+1}$ is very nearly equal to $\Delta_3 x_n$. As an approximation $\Delta_3 x_{n+1}$ may be taken equal to $\Delta_3 x_n$, or perhaps a closer value may be determined from the way the third differences

$\Delta_3 x_{n-3}$, $\Delta_3 x_{n-2}$, $\Delta_3 x_{n-1}$, and $\Delta_3 x_n$ vary. For example, in Table II it is easy to see that $\Delta_3 \sin 75^\circ$ is almost certainly $-3$. It follows from **10.20, 1, 2** that

$$3. \quad \begin{cases} \Delta_2 x_{n+1} = \Delta_3 x_{n+1} + \Delta_2 x_n, \\ \Delta_1 x_{n+1} = \Delta_2 x_{n+1} + \Delta_1 x_n, \\ x_{n+1} = \Delta_1 x_{n+1} + x_n. \end{cases}$$

After the adopted value of $\Delta_3 x_{n+1}$ has been written in its column the successive entries to the left can be written down by simple additions to the respective numbers on the line of $t_n$. For example, it is found from Table II that $\Delta_2 \sin 75^\circ = -72$, $\Delta_1 \sin 75^\circ = 262$, $\sin 75^\circ = 9659$. This is, indeed, the correct value of $\sin 75^\circ$ to four places.

Now having extrapolated approximate values of $x_{n+1}$ and $y_{n+1}$ it remains to compute $f$ and $g$ for $x = x_{n+1}$, $y = y_{n+1}$, $t = t_{n+1}$. The next step is to pass curves through the values of $f$ and $g$ for $t = t_{n+1}, t_n, t_{n-1}, \ldots$ and to compute the integrals (2). This is the precise problem that was solved in **10.30,** the only difference being that in that section the integrand was designated by $y$. On applying equation **10.30** (9) to the computation of the integrals (2), the latter give

$$4. \quad \begin{cases} x_{n+1} = x_n + h\left[f_{n+1} - \frac{1}{2}\Delta_1 f_{n+1} - \frac{1}{12}\Delta_2 f_{n+1} - \frac{1}{24}\Delta_3 f_{n+1} \ldots\right], \\ y_{n+1} = y_n + h\left[g_{n+1} - \frac{1}{2}\Delta_1 g_{n+1} - \frac{1}{12}\Delta_2 g_{n+1} - \frac{1}{24}\Delta_3 g_{n+1} \ldots\right], \end{cases}$$

where

$$5. \quad \begin{cases} f_{n+1} = f(x_{n+1}, y_{n+1}, t_{n+1}), \\ g_{n+1} = g(x_{n+1}, y_{n+1}, t_{n+1}). \end{cases}$$

The right members of (4) are known and therefore $x_{n+1}$ and $y_{n+1}$ are determined.

It will be recalled that $f_{n+1}$ and $g_{n+1}$ were computed from extrapolated values of $x_{n+1}$ and $y_{n+1}$, and hence are subject to some error. They should now be recomputed with the values of $x_{n+1}$ and $y_{n+1}$ furnished by (4). Then more nearly correct values of the entire right members of (4) are at hand and the values of $x_{n+1}$ and $y_{n+1}$ should be corrected if necessary. If the interval $h$ is small it will not generally be necessary to correct $x_{n+1}$ and $y_{n+1}$. But if they require corrections, then new values of $f_{n+1}$ and $g_{n+1}$ should be computed. In practice it is advisable to take the interval $h$ so small that one correction to $f_{n+1}$ and $g_{n+1}$ is sufficient.

After $x_{n+1}$ and $y_{n+1}$ have been obtained, values of $x$ and $y$ at $t_{n+2}$ can be found in precisely the same manner, and the process can be continued to $t = t_{n+3}, t_{n+4}, \ldots$. If the higher differences become large and irregular it is advisable to interpolate values at the mid-intervals of the last two steps and to continue with an interval half as great. On the other hand, if the higher differences become very small it is advisable to proceed with an interval twice as great as that used in the earlier part of the computation.

The foregoing, expressed in words, seems rather complicated. As a matter of fact, it goes very simply in practice, as will be shown in section **10.9.**

**10.8** The Start of the Construction of the Solution. Suppose the differential equations are again

$$1. \qquad \begin{cases} \dfrac{dx}{dt} = f(x, y, t), \\ \dfrac{dy}{dt} = g(x, y, t), \end{cases}$$

with the initial conditions $x = a$, $y = b$ at $t = 0$. Only the initial values of $x$ and $y$ are known. But it follows from (1) that the rates of change of $x$ and $y$ at $t = 0$ are $f(a, b, 0)$ and $g(a, b, 0)$ respectively. Consequently, first approximations to values of $x$ and $y$ at $t = t_1 = h$ are

$$2. \qquad \begin{cases} x_1^{(1)} = a + hf(a, b, 0), \\ y_1^{(1)} = b + hg(a, b, 0). \end{cases}$$

Now it follows from (1) that the rates of change of $x$ and $y$ at $x = x_1$, $y = y_1$, $t = t_1$ are approximately $f(x_1^{(1)}, y_1^{(1)}, t_1)$ and $g(x_1^{(1)}, y_1^{(1)}, t_1)$. These rates will be different from those at the beginning, and the average rates of change for the first interval will be nearly the average of the rates at the beginning and at the end of the interval. Therefore closer approximations than those given in (2) to the values of $x$ and $y$ at $t = t_1$ are

$$3. \qquad \begin{cases} x_1^{(2)} = a + \tfrac{1}{2}h\,[f(a, b, 0) + f(x_1^{(1)}, y_1^{(1)}, t_1)], \\ y_1^{(2)} = b + \tfrac{1}{2}h\,[g(a, b, 0) + g(x_1^{(1)}, y_1^{(1)}, t_1)]. \end{cases}$$

The process could be repeated on the first interval, but it is not advisable when the interval is taken as short as it should be.

The rates of change at the beginning of the second interval are approximately $f(x_1^{(2)}, y_1^{(2)}, t_1)$ and $g(x_1^{(2)}, y_1^{(2)}, t_1)$ respectively. Consequently, first approximations to the values of $x$ and $y$ at $t = t_2$, where $t_2 - t_1 = h$, are

$$4. \qquad \begin{cases} x_2^{(1)} = x_1^{(2)} + hf(x_1^{(2)}, y_1^{(2)}, t_1), \\ y_2^{(1)} = y_1^{(2)} + hg(x_1^{(2)}, y_1^{(2)}, t_1). \end{cases}$$

With these values of $x$ and $y$ approximate values of $f_2$ and $g_2$ are computed. Since $f_0, g_0$; $f_1, g_1$ are known, it follows that $\Delta_1 f_2$, $\Delta_1 g_2$; $\Delta_2 f_2$, and $\Delta_2 g_2$ are also known. Hence equations (4) of **10.7**, for $n + 1 = 2$, can be used, with the exception of the last terms in the right members, for the computation of $x_2$ and $y_2$.

At this stage of work $x_0 = a$, $y_0 = b$; $x_1, y_1$; $x_2, y_2$ are known, the first pair exactly and the last two pairs with considerable approximation. After $f_2$ and $g_2$ have been computed, $x_1$ and $y_1$ can be corrected by **10.31** for $n = 1$. Then approximate values of $x_3$ and $y_3$ can be extrapolated by the method explained in the preceding section, after which approximate values of $f_3$ and $g_3$ can be computed. With these values and the corresponding difference functions, $x_2$ and $y_2$ can be corrected by using **10.31.** Then after correcting all the corresponding differences of all the functions, the solution is fully started and proceeds by the method given in the preceding section.

**10.9** Numerical Illustration. In this section a numerical problem will be treated which will illustrate both the steps which must be taken and also the method of

arranging the work. A convenient arrangement of the computation which preserves a complete record of all the numerical work is very important.

Suppose the differential equation is

$$1. \qquad \begin{cases} \dfrac{d^2x}{dt^2} = -(1+\kappa^2)x + 2\kappa^2x^3, \\ x = 0, \dfrac{dx}{dt} = 1 \text{ at } t = 0. \end{cases}$$

The problem of the motion of a simple pendulum takes this form when expressed in suitable variables. This problem is chosen here because it has an actual physical interpretation, because it can be integrated otherwise so as to express $t$ in terms of $x$, and because it will illustrate sufficiently the processes which have been explained.

Equation (1) will first be integrated so as to express $t$ in terms of $x$. On multiplying both sides of (1) by $2\dfrac{dx}{dt}$ and integrating, it is found that the integral which satisfies the initial conditions is

$$2. \qquad \left(\frac{dx}{dt}\right)^2 = (1-x^2)\,(1-\kappa^2x^2).$$

On separating the variables this equation gives

$$3. \qquad t = \int_0^x \frac{dx}{\sqrt{(1-x^2)\,(1-\kappa^2x^2)}}.$$

Suppose $\kappa^2 < 1$ and that the upper limit $x$ does not exceed unity. Then

$$4. \qquad \frac{1}{\sqrt{1-\kappa^2x^2}} = 1 + \frac{1}{2}\kappa^2x^2 + \frac{3}{8}\kappa^4x^4 + \frac{5}{16}\kappa^6x^6 + \ldots.$$

where the right member is a converging series. On substituting (4) into (3) and integrating, it is found that

$$5. \quad t = \sin^{-1}x + \tfrac{1}{4}\left[-x\sqrt{1-x^2} + \sin^{-1}x\right]\kappa^2 + \tfrac{3}{8}\left[-x^3\sqrt{1-x^2} - \tfrac{3}{4}x(1-x^2)^{\frac{3}{2}}\right.$$
$$\left. + \tfrac{3}{8}x\sqrt{1-x^2} + \tfrac{3}{8}\sin^{-1}x\right]\kappa^4 + \ldots\ldots\ldots.$$

When $x = 1$ this integral becomes

$$6. \qquad T = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2\kappa^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2\kappa^4 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\kappa^6 + \ldots\right].$$

Equation (5) gives $t$ for any value of $x$ between $-1$ and $+1$. But the problem is to determine $x$ in terms of $t$. Of course, if a table is constructed giving $t$ for many values of $x$, it may be used inversely to obtain the value of $x$ corresponding to any value of $t$. The labor involved is very great. When $\kappa^2$ is given numerically it is simpler to compute the integral (3) by the method of **10.1** or **10.3**.

In mathematical terms, $t$ is an elliptical integral of $x$ of the first kind, and the inverse function, that is, $x$ as a function of $t$, is the sine-amplitude function, which has the real period $4T$.

Suppose $\kappa^2 = \frac{1}{2}$ and let $y = \frac{dx}{dt}$. Then equation (1) is equivalent to the two equations

$$7. \qquad \begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\frac{3}{2}x + x^3, \end{cases}$$

which are of the form **10.50** (1), where

$$8. \qquad \begin{cases} f = y, \\ g = -\frac{3}{2}x + x^3, \end{cases}$$

and $x = 0$, $y = 1$ at $t = 0$.

The first step is to determine the interval which is to be used in the start of the solution. No general rule can be given. The larger $f_0$ and $g_0$ the smaller must the interval be taken. A fairly good rule is in general to take $h$ so small that $hf_0$ and $hg_0$ shall not be greater than 1000 times the permissible error in the results. In the present instance we may take $h = 0.1$.

First approximations to $x$ and $y$ at $t = 0.1$ are found from the initial conditions and equations **10.8** (2) to be

$$9. \qquad \begin{cases} x_1^{(1)} = 0 + \frac{1}{10}\,1 = 0.1000, \\ y_1^{(1)} = 1 + \frac{1}{10}\,0 = 1.0000. \end{cases}$$

It follows from (8) and these values of $x_1$ and $y_1$ that

$$10. \qquad \begin{cases} f(x_1^{(1)}, y_1^{(1)}, t_1) = 1.0000, \\ g(x_1^{(1)}, y_1^{(1)}, t_1) = -0.1490. \end{cases}$$

Hence the more nearly correct values of $x_1$ and $y_1$, which are given by **10.8** (3), are

$$11. \qquad \begin{cases} x_1^{(2)} = 0 + \frac{0.1}{2}\,[1.0000 + 1.0000] = 0.1000, \\ y_1^{(2)} = 1 + \frac{0.1}{2}\,[0.0000 - 0.1490] = 0.9925. \end{cases}$$

Since in this particular problem $x = \int y\,dt$, it is not necessary to compute both $f$ and $g$ by the exact process explained in section **10.8**, for after $y$ has been determined $x$ is given by the integral. It follows from (7), (8), (10), and (11) that a first approximation to the value of $y$ at $t = t_2 = 0.2$ is

$$12. \qquad y_2^{(1)} = .0025 - \frac{1}{10}.1490 = .9776.$$

With the values of $y$ at 0, .1, .2 given by the initial conditions and in equations (9) and (12), the first trial $y$-table is constructed as follows:

First Trial $y$-Table

| $t$ | $y$ | $\Delta_1 y$ | $\Delta_2 y$ |
|---|---|---|---|
| 0 | 1.0000 | | |
| .1 | .9925 | −.0075 | |
| .2 | .9776 | −.0149 | −.0074 |

Since $y = f$ it now follows from the first equations of (11) and **10.7** (4) for $n = 1$ that an approximate value of $x_2$ is

13. $$x_2^{(1)} = 0.1000 + \frac{1}{10}\left[.9776 + \frac{1}{2}.0149 + \frac{1}{12}.0074\right] = .1986.$$

With this value of $x_2$ it is found from the second of (8) that $g_2 = .2901$. Then the first trial $g$-table constructed from the values of $g$ at $t = 0, 0.1, 0.2$, is:

First Trial $g$-Table

| $t$ | $g$ | $\Delta_1 g$ | $\Delta_2 g$ |
|---|---|---|---|
| 0 | .0000 | | |
| .1 | −.1490 | −.1490 | |
| .2 | −.2901 | −.1411 | +.0079 |

Then the second equation of **10.7** (4) gives for $n = 1$ the more nearly correct value of $y_2$,

14. $$y_2 = .9925 + \frac{1}{10}\left[-.2901 + \frac{1}{12}.1411 - \frac{1}{12}.0079\right] = .9705.$$

This value of $y_2$ should replace the last entry in the first trial $y$-table. When this is done it is found that $\Delta_1 y_2 = -.0220$, $\Delta_2 y_2 = -.0145$. Then the first equation of **10.7** (4) gives

15. $$x_2 = .1000 + \frac{1}{10}\left[.9705 + \frac{1}{2}.0220 + \frac{1}{12}.0145\right] = .1983.$$

The computation is now well started although $x_1$, $y_1$, $x_2$, and $y_2$ are still subject to slight errors. The values of $x_1$ and $y_1$ can be corrected by applying **10.31** for $n = 1$. It is necessary first to compute a more nearly correct value of $g_2$ by using the value of $x_2$ given in (15). The result is $g_2 = -.2896$, $\Delta_1 g_2 = -.1406$, $\Delta_2 g_2 = +.0084$. Then the second equation of **10.7** (4) gives

16. $$y_2 = .9925 + \frac{1}{10}\left[-.2896 + \frac{1}{2}.1406 - \frac{1}{12}.0084\right] = .9705,$$

agreeing with (14). This value of $y_2$ is therefore essentially correct. An application of **10.31** then gives

17. $$x_1 = .0000 + \frac{1}{10}\left[.9705 + \frac{3}{2}.0220 - \frac{5}{12}.0145\right] = .0997,$$

after which it is found that $g_1 = -.1486$, $\Delta_1 g_1 = -.1486$. Now the first trial $y$-table can be corrected by using the value of $y_2$ given in (14). The result is:

Second Trial $y$-Table

| $t$ | $y$ | $\Delta_1 y$ | $\Delta_2 y$ |
|---|---|---|---|
| 0 | 1.0000 | | |
| .1 | .9925 | −.0075 | |
| .2 | .9705 | −.0220 | −.0145 |

In order to correct $x_2$ and $y_2$ by the same method, which is the most convenient one to follow, it is necessary first to obtain approximate values of $g_3$ and $y_3$. The trial $g$-table can be corrected by computing $g$ with the values of $x$ given by (17) and (15). Then the line for $g_3$ can be extrapolated. The results are:

Second Trial $g$-Table

| $t$ | $g$ | $\Delta_1 g$ | $\Delta_2 g$ |
|---|---|---|---|
| 0 | .0000 | | |
| .1 | −.1486 | −.1486 | |
| .2 | −.2896 | −.1410 | +.0076 |
| .3 | −.4230 | −.1334 | +.0076 |

Then the second equation of **10.7** (4) gives for $n = 2$,

18. $$y_3 = .9705 + \frac{1}{10}\left[-.4230 + \frac{1}{2}.1334 - \frac{1}{12}.0076\right] = .9348.$$

When this is added to the second trial $y$-table, it is found that

19. $$y_3 = .9348,\ \Delta_1 y_3 = -.0357,\ \Delta_2 y_3 = -.0137,\ \Delta_3 y_3 = +.0008.$$

Now $x_2$ and $y_2$ can be corrected by applying **10.31** to these numbers and those in the last line of the second trial $g$-table. The results are

20. $$\begin{cases} x_2 = .0997 + \frac{1}{10}\left[.9348 + \frac{3}{2}.0357 - \frac{5}{12}.0137 + \frac{1}{24}.0008\right] = .1980, \\ y_2 = .9925 + \frac{1}{10}\left[-.4230 + \frac{3}{2}.1334 + \frac{5}{12}.0076\right] = .9705. \end{cases}$$

The preliminary work is finished and $x$ and $y$ have been determined for $t = 0$, .1, and .2 with an error of probably not more than one unit in the last place. As the process is read over it may seem somewhat complicated, but this is largely because on the printed page preliminary values of the unknown quantities can not be erased and replaced by more nearly correct ones. As a matter of fact, the

first steps are very simple and can be carried out in practice in a few minutes if the chosen time-interval is not too great.

The problem now reduces to simple routine. There are an $x$-table, a $y$-table (which in this problem serves also as an $f$-table), a $g$-table, and a schedule for computing $g$. It is advisable to use large sheets so that all the computations except the schedule for computing $g$ can be kept side by side on the same sheet. The process consists of six steps: (1) Extrapolate a value of $g_{n+1}$ and its differences in the $g$-table; (2) compute $y_{n+1}$ by the second equation of **10.7** (4); (3) enter the result in the $y$-table and write down the differences; (4) use these results to compute $x_{n+1}$ by the first equation of **10.7** (4); (5) with this value of $x_{n+1}$ compute $g_{n+1}$ by the $g$-computation schedule; and (6) correct the extrapolated value of $g_{n+1}$ in the $g$-table.

Usually the correction to $g_{n+1}$ will not be great enough to require a sensible correction to $y_{n+1}$. But if a correction is required, it should, of course, be made. It follows from the integration formulas **10.7** (4) and the way that the difference functions are formed that an error $\epsilon$ in $g_{n+1}$ produces the error $\frac{3}{8}h\epsilon$ in $y_{n+1}$, and the corresponding error in $x_{n+1}$ is $\frac{9}{64}h^2\epsilon$. It is never advisable to use so large a value of $h$ that the error in $x_{n+1}$ is appreciable. On the other hand, if the differences in the $g$-table and the $y$-table become so small that the second differences are insensible the interval may be doubled.

The following tables show the results of the computations in this problem reduced from five to four places.

Final $x$-Table

| $t$ | $x$ | $\Delta_1 x$ | $\Delta_2 x$ | $\Delta_3 x$ |
|---|---|---|---|---|
| 0 | .0000 | | | |
| .1 | .0997 | .0997 | | |
| .2 | .1980 | .0983 | −.0014 | |
| .3 | .2934 | .0954 | −.0029 | −.0015 |
| .4 | .3847 | .0913 | −.0041 | −.0012 |
| .5 | .4708 | .0861 | −.0052 | −.0011 |
| .6 | .5508 | .0800 | −.0061 | −.0009 |
| .7 | .6243 | .0735 | −.0065 | −.0004 |
| .8 | .6909 | .0666 | −.0069 | −.0004 |
| .9 | .7505 | .0596 | −.0070 | −.0001 |
| 1.0 | .8030 | .0525 | −.0071 | −.0001 |
| 1.1 | .8486 | .0456 | −.0069 | +.0002 |
| 1.2 | .8877 | .0391 | −.0065 | +.0004 |
| 1.3 | .9205 | .0328 | −.0063 | +.0002 |
| 1.4 | .9472 | .0267 | −.0061 | +.0002 |
| 1.5 | .9682 | .0210 | −.0057 | +.0004 |
| 1.6 | .9837 | .0155 | −.0055 | +.0002 |
| 1.7 | .9940 | .0103 | −.0052 | +.0003 |
| 1.8 | .9993 | .0053 | −.0050 | +.0002 |
| 1.9 | .9995 | .0002 | −.0051 | −.0001 |

Final $y$-Table

| $t$ | $y$ | $\Delta_1 y$ | $\Delta_2 y$ | $\Delta_3 y$ |
|---|---|---|---|---|
| 0 | 1.0000 | | | |
| .1 | .9925 | −.0075 | | |
| .2 | .9705 | −.0220 | −.0145 | |
| .3 | .9352 | −.0353 | −.0133 | +.0012 |
| .4 | .8882 | −.0470 | −.0117 | +.0016 |
| .5 | .8320 | −.0562 | −.0092 | +.0025 |
| .6 | .7687 | −.0633 | −.0071 | +.0019 |
| .7 | .7009 | −.0678 | −.0045 | +.0016 |
| .8 | .6308 | −.0701 | −.0023 | +.0022 |
| .9 | .5602 | −.0706 | −.0005 | +.0008 |
| 1.0 | .4906 | −.0696 | +.0010 | +.0015 |
| 1.1 | .4231 | −.0675 | +.0021 | +.0011 |
| 1.2 | .3584 | −.0647 | +.0028 | +.0007 |
| 1.3 | .2968 | −.0616 | +.0031 | +.0003 |
| 1.4 | .2382 | −.0586 | +.0030 | −.0001 |
| 1.5 | .1824 | −.0558 | +.0028 | −.0002 |
| 1.6 | .1290 | −.0534 | +.0024 | −.0004 |
| 1.7 | .0775 | −.0515 | +.0019 | −.0005 |
| 1.8 | .0271 | −.0504 | +.0011 | −.0008 |
| 1.9 | −.0230 | −.0501 | +.0003 | −.0008 |

Final $g$-Schedule

| $t$ | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|---|---|---|---|
| $\log x$ | 8.9989 | 9.2967 | 9.4675 | 9.5851 | 9.6728 | 9.7410 | 9.7954 | 9.8394 | 9.8753 |
| $\log x^3$ | 6.9967 | 7.8901 | 8.4025 | 8.7553 | 9.0184 | 9.2230 | 9.3862 | 9.5182 | 9.6259 |
| $3x$ | .2092 | .5941 | .8802 | 1.1541 | 1.4124 | 1.6524 | 1.8729 | 2.0727 | 2.2515 |
| $-\frac{3}{2}x$ | −.1496 | −.2970 | −.4401 | −.5770 | −.7062 | −.8262 | −.9365 | −1.0364 | −1.1257 |
| $x^3$ | .0010 | .0077 | .0252 | .0569 | .1044 | .1671 | .2434 | .3298 | .4227 |
| $g$ | −.1486 | −.2893 | −.4149 | −.5201 | −.6018 | −.6591 | −.6931 | −.7066 | −.7030 |

Final $g$-Table

| $t$ | $g$ | $\Delta_1 g$ | $\Delta_2 g$ | $\Delta_3 g$ |
|---|---|---|---|---|
| 0 | .0000 | | | |
| .1 | −.1486 | −.1486 | | |
| .2 | −.2893 | −.1407 | +.0079 | |
| .3 | −.4149 | −.1256 | +.0151 | +.0072 |
| .4 | −.5201 | −.1052 | +.0204 | +.0053 |
| .5 | −.6018 | −.0817 | +.0235 | +.0031 |
| .6 | −.6591 | −.0573 | +.0244 | +.0009 |
| .7 | −.6931 | −.0340 | +.0233 | −.0011 |
| .8 | −.7066 | −.0135 | +.0205 | −.0028 |
| .9 | −.7030 | +.0036 | +.0171 | −.0034 |
| 1.0 | −.6867 | +.0163 | +.0127 | −.0044 |
| 1.1 | −.6618 | +.0249 | +.0086 | −.0041 |
| 1.2 | −.6320 | +.0298 | +.0049 | −.0037 |
| 1.3 | −.6008 | +.0312 | +.0014 | −.0035 |
| 1.4 | −.5710 | +.0298 | −.0014 | −.0028 |
| 1.5 | −.5447 | +.0263 | −.0035 | −.0021 |
| 1.6 | −.5236 | +.0211 | −.0052 | −.0017 |
| 1.7 | −.5088 | +.0148 | −.0063 | −.0011 |
| 1.8 | −.5011 | +.0077 | −.0071 | −.0008 |
| 1.9 | −.5008 | +.0003 | −.0074 | −.0003 |

Final $g$-Schedule — *Continued*

| 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 |
|---|---|---|---|---|---|---|---|---|---|
| 9.9047 | 9.9287 | 9.9483 | 9.9640 | 9.9764 | 9.9860 | 9.9929 | 9.9974 | 9.9997 | 9.9998 |
| 9.7141 | 9.7861 | 9.8449 | 9.8920 | 9.9292 | 9.9580 | 9.9787 | 9.9922 | 9.9991 | 9.9994 |
| 2.4090 | 2.5458 | 2.6631 | 2.7615 | 2.8416 | 2.9046 | 2.9511 | 2.9820 | 2.9979 | 2.9985 |
| −1.2045 | −1.2729 | −1.3316 | −1.3807 | −1.4208 | −1.4523 | −1.4756 | −1.4910 | −1.4989 | −1.4992 |
| .5178 | .6111 | .6996 | .7799 | .8498 | .9076 | .9520 | .9822 | .9978 | .9984 |
| − .6867 | − .6618 | − .6320 | − .6008 | − .5710 | − .5447 | − .5236 | − .5088 | − .5011 | − .5008 |

As has been remarked, large sheets should be used so that the $x$, $y$, and $g$-tables can be put side by side on one sheet. Then the $t$-column need be written but once for these three tables. The $g$-schedule, which is of a different type, should be on a separate sheet.

The differential equation (1) has an integral which becomes for $\kappa^2 = \frac{1}{2}$ and $\frac{dx}{dt} = y$.

21. $$y^2 + \frac{3}{2}x^2 - \frac{1}{4}x^4 = 1,$$

and which may be used to check the computation because it must be satisfied at every step. It is found on trial that (21) is satisfied to within one unit in the fourth place by the results given in the foregoing tables for every value of $t$.

The value of $t$ for which $x = 1$ and $y = 0$ is given by (6). When $\kappa^2 = \frac{1}{2}$ it is found that $T = 1.8541$. It is found from the final $x$-table by interpolation based on first and second differences that $x$ rises to its maximum unity for almost exactly this value of $t$; and, similarly, that $y$ vanishes for this value of $t$.

# XI ELLIPTIC FUNCTIONS

By Sir George Greenhill, F. R. S.

# INTRODUCTION TO THE TABLES OF ELLIPTIC FUNCTIONS

By Sir George Greenhill

In the integral calculus, $\int \frac{dx}{\sqrt{X}}$, and more generally, $\int \frac{M + N\sqrt{X}}{P + Q\sqrt{X}}\, dx$, where $M$, $N$, $P$, $Q$ are rational algebraical functions of $x$, can always be expressed by the elementary functions of analysis, the algebraical, circular, logarithmic or hyperbolic, so long as the degree of $X$ does not exceed the second. But when $X$ is of the third or fourth degree, new functions are required, called elliptic functions, because encountered first in the attempt at the rectification of an ellipse by means of an integral.

To express an elliptic integral numerically, when required in an actual question of geometry, mechanics, or physics and electricity, the integral must be normalised to a standard form invented by Legendre before the Tables can be employed; and these Tables of the Elliptic Functions have been calculated as an extension of the usual tables of the logarithmic and circular functions of trigonometry. The reduction to a standard form of any assigned elliptic integral that arises is carried out in the procedure described in detail in a treatise on the elliptic functions.

**11.1.** Legendre's Standard Elliptic Integral of the First Kind (E. I. I) is

$$F\phi = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}} = \int_0^{x} \frac{dx}{\sqrt{(1 - x^2)(1 - \kappa^2 x^2)}} = u,$$

defining $\phi$ as the amplitude of $u$, to the modulus $\kappa$, with the notation,

$$\phi = \text{am } u$$
$$x = \sin \phi = \text{sn am } u$$

abbreviated by Gudermann to,

$$x = \text{sn } u$$
$$\cos \phi = \text{cn } u$$
$$\Delta \phi = \sqrt{(1 - \kappa^2 \sin^2 \phi)} = \Delta \text{ am } u = \text{dn } u,$$

and sn $u$, cn $u$, dn $u$ are the three elliptic functions. Their differentiations are,

$$\frac{d\phi}{du} = \Delta\phi \qquad \text{or} \quad \frac{d \text{ am } u}{du} = \text{dn } u$$

$$\frac{d \sin \phi}{du} = \cos \phi \cdot \Delta\phi \qquad \text{or} \quad \frac{d \text{ sn } u}{du} = \text{cn } u \text{ dn } u$$

$$\frac{d\cos\phi}{du} = -\sin\phi\,\Delta\phi \qquad \text{or} \quad \frac{d\,\mathrm{cn}\,u}{du} = -\,\mathrm{sn}\,u\,\mathrm{dn}\,u$$

$$\frac{d\Delta\phi}{du} = -\kappa^2\sin\phi\cos\phi \quad \text{or} \quad \frac{d\,\mathrm{dn}\,u}{du} = -\kappa^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u$$

**11.11.** The complete integral over the quadrant, $0 < \phi < \frac{\pi}{2}$, $0 < x < 1$, defines the (quarter) period, $K$,

$$K = F\frac{\pi}{2} = \int_0^{\frac{1}{2}\pi}\frac{d\phi}{\Delta\phi},$$

making

$$\mathrm{sn}\ K = 1$$
$$\mathrm{cn}\ K = 0$$
$$\mathrm{dn}\ K = \kappa'.$$

$\kappa'$ is the comodulus to $\kappa$, $\kappa^2 + \kappa'^2 = 1$, and the coperiod, $K'$, is,

$$K' = \int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{(1-\kappa'^2\sin^2\phi)}}.$$

**11.12.**

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$$
$$\mathrm{cn}^2 u + \kappa^2\,\mathrm{sn}^2 u = 1$$
$$\mathrm{dn}^2 u - \kappa^2\,\mathrm{cn}^2 u = \kappa'^2.$$
$$\mathrm{sn}\ 0 = 0, \quad \mathrm{cn}\ 0 = \mathrm{dn}, \qquad 0 = 1.$$
$$\mathrm{sn}\ K = 1, \quad \mathrm{cn}\ K = 0, \qquad \mathrm{dn}\ K = \kappa'.$$

**11.13.** Legendre has calculated for every degree of $\theta$, the modular angle, $\kappa = \sin\theta$, the value of $F\phi$ for every degree in the quadrant of the amplitude $\phi$, and tabulated them in his Table IX, Fonctions elliptiques, t. II, $90 \times 90 = 8100$ entries.

But in this new arrangement of the Table, we take $u = F\phi$ as the independent variable of equal steps, and divide it into 90 degrees of a quadrant $K$, putting

$$u = eK = \frac{r^\circ}{90^\circ}K, \qquad r^\circ = 90^\circ e.$$

As in the ordinary trigonometrical tables, the degrees of $r$ run down the left of the page from $0^\circ$ to $45^\circ$, and rise up again on the right from $45^\circ$ to $90^\circ$. Then columns II, III, X, XI are the equivalent of Legendre's Table of $F\phi$ and $\phi$, but rearranged so that $F\phi$ proceeds by equal increments $1^\circ$ in $r^\circ$, and the increments in $\phi$ are unequal, whereas Legendre took equal increments of $\phi$ giving unequal increments in $u = F\phi$.

The reason of this rearrangement was the great advance made in elliptic function theory when Abel pointed out that $F\phi$ was of the nature of an inverse function, as it would be in a degenerate circular integral with zero modular angle. On Abel's recommendation, the notation is reversed, and $\phi$ is to be

considered a function of $u$, denoted already by $\phi = \text{am } u$, instead of looking at $u$, in Legendre's manner, as a function, $F\phi$, of $\phi$. Jacobi adopted the idea in his Fundamenta nova, and employs the elliptic functions

$$\sin\phi = \sin \text{am } u, \qquad \cos\phi = \cos \text{am } u, \qquad \Delta\phi = \Delta \text{am } u,$$

single-valued, uniform, periodic functions of the argument $u$, with (quarter) period $K$, as $\phi$ grows from 0 to $\frac{1}{2}\pi$. Gudermann abbreviated this notation to the one employed usually today.

**11.2.** The E. I. I is encountered in its simplest form, not as the elliptic arc, but in the expression of the time in the pendulum motion of finite oscillation, unrestricted to the small invisible motion of elementary treatment.

The compound pendulum, as of a clock, is replaced by its two equivalent particles, one at $O$ in the centre of suspension, and the other at the centre of oscillation, $P$; the particles are adjusted so as to have the same total weight as the pendulum, the same centre of gravity at $G$, and the same moment of inertia about $G$ or $O$; the two particles, if rigidly connected, are then the kinetic equivalent of the compound pendulum and move in the same way in the same field of force (Maxwell, Matter and Motion, CXXI).

Putting $OP = l$, called the simple equivalent pendulum length, and $P$ starting from rest at $B$, in Figure 1, the particle $P$ will move in the circular arc $BAB'$ as if sliding down a smooth curve; and $P$ will acquire the same velocity as if it fell vertically $KP = ND$; this is all the dynamical theory required.

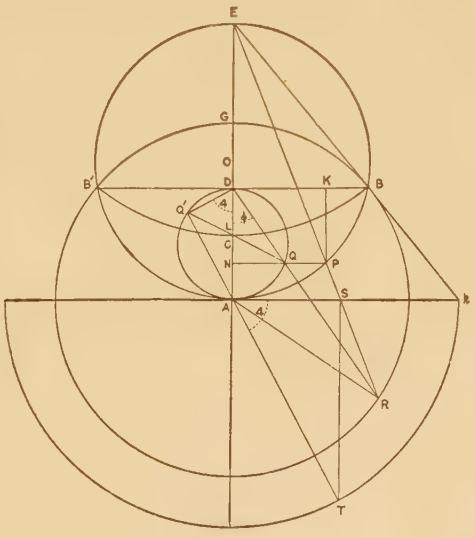

FIG. 1

$$(\text{velocity of } P)^2 = 2g \cdot KP,$$

$$(\text{velocity of } N)^2 = 2g \cdot ND \cdot \sin^2 AOP$$

$$= 2g \cdot ND \cdot \frac{NP^2}{OP^2} = \frac{g2}{l^2} \cdot ND \cdot NA \cdot NE,$$

and with $AD = h$, $AN = y$, $ND = h - y$, $AE = 2l$, $NE = 2l - y$,

$$\left(\frac{dy}{dt}\right)^2 = \frac{2g}{l^2}(hy - y^2)(2l - y) = \frac{2g}{l^2} Y,$$

where $Y$ is a cubic in $y$. Then $t$ is given by an elliptic integral of the form $\int \frac{dy}{\sqrt{Y}}$. This integral is normalised to Legendre's standard form of his E. I. I by putting $y = h \sin^2\phi$, making $AOQ = \phi$, $h - y = h \cos^2\phi$, $2l - y = 2l(1 - \kappa^2 \sin^2\phi)$,

$$\kappa^2 = \frac{h}{2l} = \frac{AD}{AE} = \sin^2 AEB.$$

$\kappa$ is called the modulus, $AEB$ the modular angle which Legendre denoted by $\theta$; $\sqrt{(1 - \kappa^2 \sin^2\phi)}$ he denoted by $\Delta\phi$.

With $g = ln^2$, and reckoning the time $t$ from $A$, this makes

$$nt = \int_0^{\phi} \frac{d\phi}{\Delta\phi} = F\phi,$$

in Legendre's notation. Then the angle $\phi$ is called the amplitude of $nt$, to be denoted am $nt$, the particle $P$ starting up from $A$ at time $t = 0$; and with $u = nt$,

$$\operatorname{sn} u = \frac{AP}{AB} = \frac{AQ}{AD} \qquad \operatorname{sn}^2 u = \frac{AN}{AD}$$

$$\operatorname{cn} u = \frac{DQ}{AD} \qquad \operatorname{cn}^2 u = \frac{PK}{AD}$$

$$\operatorname{dn} u = \frac{EP}{EA} \qquad \operatorname{dn}^2 u = \frac{NE}{AE}$$

Velocity of $P = n \cdot AB \cdot \operatorname{cn} u = \sqrt{BP \cdot PB'}$, with an oscillation beat of $T$ seconds in $u = eK$, $e = 2t/T$.

**11.21.** The numerical values of sn, cn, dn, tn $(u, \kappa)$ are taken from a table to modulus $\kappa = \sin$ (modular angle, $\theta$) by means of the functions $Dr$, $Ar$, $Br$, $Cr$, in columns V, VI, VII, VIII, by the quotients,

$$\sqrt{\kappa'} \operatorname{sn} eK = \frac{A}{D}$$

$$\operatorname{cn} eK = \frac{B}{D}$$

$$\frac{\operatorname{dn} eK}{\sqrt{\kappa'}} = \frac{C}{D}$$

$$\sqrt{\kappa'} \operatorname{tn} eK = \frac{A}{B}$$

$$r^\circ = 90^\circ e$$

$$u = eK.$$

These $D$, $A$, $B$, $C$ are the Theta Functions of Jacobi, normalised, defined by

$$D(r) = \frac{\Theta u}{\Theta 0}, \qquad A(r) = \frac{Hu}{HK},$$

$$B(r) = A(90^\circ - r) \qquad C(r) = D(90^\circ - r).$$

They were calculated from the Fourier series of angles proceeding by multiples of $r^\circ$, and powers of $q$ as coefficients, defined by

$$q = e^{-\pi \frac{k'}{k}}$$

$$\Theta u = 1 - 2q \cos 2r + 2q^4 \cos 4r - 2q^9 \cos 6r + \ldots.$$

$$Hu = 2q^{\frac{1}{4}} \sin r - 2q^{\frac{9}{4}} \sin 3r + 2q^{\frac{25}{4}} \sin 5r - \ldots.$$

**11.3.** The Elliptic Integral of the Second Kind (E. I. II) arose first historically in the rectification of the ellipse, hence the name. With $BOP = \phi$ in Figure 2, the minor eccentric angle of $P$, and $s$ the arc $BP$ from $B$ to $P$ at $x = a \sin \phi$, $y = b \cos \phi$,

$$\frac{ds}{d\phi} = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} = a\Delta(\phi, \kappa),$$

to the modulus $\kappa$, the eccentricity of the ellipse. Then $s = a\,E\phi$, where $\int_0^\phi \Delta\phi \cdot d\phi$ is denoted by $E\phi$ in Legendre's notation of his standard E. I. II; it is tabulated in his Table IX alongside of $F\phi$ for every degree of the modular angle $\theta$, and to every degree in the quadrant of the amplitude $\phi$.

But it is not possible to make the inversion and express $\phi$ as a single-valued function of $E\phi$.

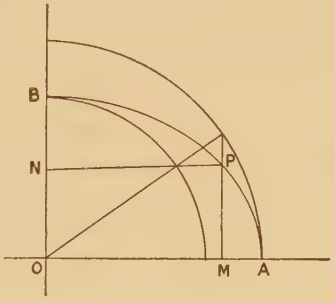


FIG. 2

**11.31.** The E. I. II, $E\phi$, arises also in the expression of the time, $t$, in the oscillation of a particle, $P$, on the arc of a parabola, as $F\phi$ was required on the arc of a circle. Starting from $B$ along the parabola $BAB'$, Figure 3, and with $AO = h$, $OB = b$, $BOQ = \phi$, $AN = y = h \cos^2 \phi$, $NP = x = b \cos \phi$ and with $OS = 2h = b \tan \alpha$, $OA' = SB = b \sec \alpha$, the parabola cutting the horizontal at $B$ at an angle $\alpha$, the modular angle, $BRA'B'$ is a semi-ellipse, with focus at $S$, and eccentricity $\kappa = \sin \alpha$.

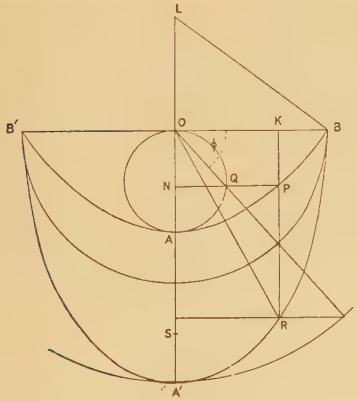


FIG. 3

$$\text{(Velocity of } P)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= (b^2 \cos^2 \phi + 4h^2 \sin^2 \phi \cos^2 \phi) \left(\frac{d\phi}{dt}\right)^2$$

$$= a^2(1 - \sin^2 \alpha \sin^2 \phi) \cos^2 \phi \left(\frac{d\phi}{dt}\right)^2 = 2gy = 2gh \cos^2 \phi$$

$$= V^2 \cos^2 \phi,$$

if $V$ denotes the velocity of $P$ at $A$, and $OA' = a$. Then with $s$ the elliptic arc $BR$,

$$V \frac{dt}{d\phi} = a\Delta\phi = a \frac{ds}{d\phi}, \quad Vt = s,$$

and so the point $R$ moves round the ellipse with constant velocity $V$, and accompanies the point $P$ on the same vertical, oscillating on the parabola from $B$ to $B'$.

In the analogous case of the circular pendulum, the time $t$ would be given by the arc of an Elastica, in Kirchhoff's Kinetic Analogue, and this can be placed as a bow on Figure 1, with the cord along $AE$ and vertex at $B$.

Legendre has shown also how in the oscillation of $R$ on the semi-ellipse $BRB'$ in a gravity field the time $t$ is expressible by elliptic integrals, two of the first and two of the second kind, to complementary modulus (Fonctions elliptiques, I, p. 183).

**11.32.** In these tables, $E\phi$ is replaced by the columns IV, IX, of $E(r)$ and $G(r) = E(90 - r)$, defined, in Jacobi's notation, by

$$E(r) = \operatorname{zn} eK = E\phi - eE$$
$$G(r) = \operatorname{zn}(1 - e)K, \quad r = 90e.$$

This is the periodic part of $E\phi$ after the secular term $eE = \frac{E}{K}u$ has been set aside, $E$ denoting the complete E. I. II,

$$E = E\tfrac{1}{2}\pi = \int^{\frac{1}{2}\pi} \Delta\phi \cdot d\phi.$$

The function zn $u$, or $Zu$ in Jacobi's notation, or $E(r)$ in our notation, is calculated from the series,

$$Er = Zu = \frac{\pi}{K}\sum_{m=1}^{\infty} \frac{\sin 2mr}{\sinh m\pi \frac{K'}{K}} = \frac{2\pi}{K}\sum_{m=1}^{\infty} (q^m + q^{3m} + q^{5m} + \ldots .) \sin 2mr.$$

This completes the explanation of the twelve columns of the tables.

**11.4.** The Double Periodicity of the Elliptic Functions.

This can be visualised in pendulum motion if gravity is supposed reversed suddenly at $B$ (Figure 1) the end of a swing; as if by the addition of a weight to bring the centre of gravity above $O$, or by the movement of a weight, as in the metronome. The point $P$ then oscillates on the arc $BEB'$, and beats the elliptic function to the complementary modulus $\kappa'$, as if in imaginary time, to imaginary argument $nti = fK'i$: and it reaches $P'$ on $AX$ produced, where $\tan AEP' = \tan AEB \cdot \operatorname{cn}(nt'i, \kappa)$, or $\tan EAP' = \tan EAB \cdot \operatorname{cn}(nt', \kappa')$; or with $\mathrm{nt}' = v$, $DR' = DB \cdot \operatorname{cn}(iv, \kappa')$, $DR = DB \cdot \operatorname{cn}(v, \kappa')$, with $DR \cdot DR' = DB^2$, $EP'$ crossing $DB$ in $R'$.

$$\operatorname{cn}(iv, \kappa) = \frac{1}{\operatorname{cn}(v, \kappa')}$$
$$\operatorname{sn}(iv, \kappa) = \frac{i \operatorname{sn}(v, \kappa')}{\operatorname{cn}(v, \kappa')} = i \operatorname{tn}(v, \kappa')$$
$$\operatorname{dn}(iv, \kappa) = \frac{\operatorname{dn}(v, \kappa')}{\operatorname{cn}(v, \kappa')} = \frac{1}{\operatorname{sn}(K' - v, \kappa')}$$

where $K'$ denotes the complementary (quarter) period to comodulus $\kappa'$.

If $m$, $m'$ are any integers, positive or negative, including 0,

$$\operatorname{sn}(u + 4mK + 2m'iK') = \operatorname{sn} u$$
$$\operatorname{cn}[u + 4mK + 2m'(K + iK')] = \operatorname{cn} u$$
$$\operatorname{dn}(u + 2mK + 4m'iK') = \operatorname{dn} u$$

**11.41.** The Addition Theorem of the Elliptic Functions.

$$\operatorname{sn}(u \pm v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{cn}(v \pm u) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{dn}(v \pm u) = \frac{\operatorname{dn} u \operatorname{dn} v \mp \kappa^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

**11.42.** Coamplitude Formulas, with $v = \pm K$,

$$\operatorname{sn}(K-u) = \frac{\operatorname{cn} u}{\operatorname{dn} u} = \operatorname{sn}(K+u)$$

$$\operatorname{cn}(K-u) = \frac{\kappa' \operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{cn}(K+u) = -\frac{\kappa' \operatorname{sn} u}{\operatorname{dn} u}$$

$$\operatorname{dn}(K-u) = \frac{\kappa'}{\operatorname{dn} u} = \operatorname{dn}(K+u)$$

$$\operatorname{tn}(K-u) = \frac{1}{\kappa' \operatorname{tn} u} \qquad \operatorname{tn}(K+u) = -\frac{1}{\kappa' \operatorname{tn} u}$$

**11.43.** Legendre's Addition Formula for his E. I. II,

$$E\phi = \int \Delta\phi \cdot d\phi = \int \operatorname{dn}^2 u \cdot du, \quad \phi = \int \operatorname{dn} u \cdot du = \operatorname{am} u.$$

$$E\phi + E\psi - E\sigma = \kappa^2 \sin\phi \sin\psi \sin\sigma, \ \psi = \operatorname{am} v, \ \sigma = \operatorname{am}(v+u)$$

or, in Jacobi's notation,

$$\operatorname{zn} u + \operatorname{zn} v - \operatorname{zn}(u+v) = \kappa^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(v+u),$$

the secular part cancelling.

Another form of the Addition Theorem for Legendre's E. I. II,

$$E\sigma - E\theta - 2E\psi = \frac{-2\kappa^2 \sin\psi \cos\psi \Delta\psi \sin^2\phi}{1 - \kappa^2 \sin^2\phi \sin^2\psi}, \ \theta = \operatorname{am}(v-u)$$

or, in Jacobi's notation,

$$\operatorname{zn}(v+u) + \operatorname{zn}(v-u) - 2\operatorname{zn} v = \frac{-2\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

**11.5.** The Elliptic Integral of the Third Kind (E. I. III) is given by the next integration with respect to $u$, and introduces Jacobi's Theta Function, $\Theta u$, defined by,

$$\frac{d \log \Theta u}{du} = Zu = \operatorname{zn} u$$

$$\frac{\Theta u}{\Theta o} = \exp. \int_o \operatorname{zn} u \cdot du.$$

Integrating then with respect to $u$,

$$\log \Theta(v+u) - \log \Theta(v-u) - 2u \operatorname{zn} v = \int_o \frac{-2\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} du,$$

and this integral is Jacobi's standard form of the E. I. III, and is denoted by $-2\Pi(u, v)$; thus,

$$\Pi(u, v) = \int \frac{\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} du = u \operatorname{zn} v + \tfrac{1}{2} \log \frac{\Theta(v-u)}{\Theta(v+u)}.$$

Jacobi's Eta Function, $Hv$, is defined by

$$\frac{Hv}{\Theta v} = \sqrt{\kappa} \operatorname{sn} v,$$

and then

$$\frac{d \log Hv}{dv} = \frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v} + \operatorname{zn} v, \text{ denoted by } \operatorname{zs} v;$$

so that

$$\int_0 \frac{\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v} du}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} = u \frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v} + \Pi(u, v)$$

$$= u \operatorname{zs} v + \frac{1}{2} \log \frac{\Theta(v - u)}{\Theta(v + u)}$$

$$= \frac{1}{2} \log \frac{\Theta(v - u)}{\Theta(v + u)} e^{2u \cdot \operatorname{zs} v}$$

This gives Legendre's standard E. I. III,

$$\int \frac{M}{1 + n \sin^2 \phi} \frac{d\phi}{\Delta\phi},$$

where we put $n = -\kappa^2 \operatorname{sn}^2 v = -\kappa^2 \sin^2 \psi$,

$$M^2 = -\left(1 + \frac{\kappa^2}{n}\right)(1 + n) = \frac{\cos^2 \psi \Delta^2 \psi}{\sin^2 \psi} = \frac{\operatorname{cn}^2 v \operatorname{dn}^2 v}{\operatorname{sn}^2 v};$$

the normalising multiplier, $M$.

The E. I. III arises in the dynamics of the gyroscope, top, spherical pendulum, and in Poinsot's herpolhode. It can be visualized in the solid angle of a slant cone, or in the perimeter of the reciprocal cone, a sphero-conic, or in the magnetic potential of the circular base.

**11.51.** We arrive here at the definitions of the functions in the tables. Jacobi's $\Theta u$ and $\mathrm{H}u$ are normalised by the divisors $\Theta o$ and $\mathrm{H}K$, and with $r = 90e$,

$$D(r) \text{ denotes } \frac{\Theta eK}{\Theta K}, \qquad A(r) \text{ denotes } \frac{\mathrm{H}eK}{\mathrm{H}K}$$

while $B(r) = A(90 - r)$, $C(r) = D(90 - r)$, and $B(o) = A(90) = D(o) = C(90) = 1$, $\mathrm{C}(o) = D(90) = \frac{1}{\sqrt{\kappa}}$.

Then in the former definitions,

$$\frac{A(r)}{D(r)} = \frac{A(90)}{D(90)} \operatorname{sn} u = \sqrt{\kappa'} \operatorname{sn} eK$$

$$\frac{B(r)}{D(r)} = \frac{B(o)}{D(o)} \operatorname{cn} u = \operatorname{cn} eK$$

$$\frac{C(r)}{D(r)} = \frac{C(o)}{D(o)} \operatorname{dn} u = \frac{\operatorname{dn} eK}{\sqrt{\kappa'}}.$$

Then, with $u = eK$, $v = fK$, $r = 90e$, $s = 90f$,

$$(u, v) = eK \operatorname{zn} fK + \frac{1}{2} \log \frac{\Theta(f - e) K}{\Theta(f + e) K}$$

$$= eK E(s) + \frac{1}{2} \log \frac{D(s - r)}{D(s + r)}$$

$$\operatorname{zn} fK = E(s), \qquad \operatorname{zn}(1 - f) K = E(90 - s) = G(s).$$

The Jacobian multiplication relations of his theta functions can then be rewritten

$$D(r+s)D(r-s) = D^2rD^2s - \tan^2\theta A^2rA^2s,$$
$$A(r+s)A(r-s) = A^2rD^2s - D^2rA^2s,$$
$$B(r+s)B(r-s) = B^2rB^2s - A^2rA^2s.$$

But unfortunately for the physical applications the number $s$ proves usually to be imaginary or complex, and Jacobi's expression is useless; Legendre calls this the circular form of the E. I. III, the logarithmic or hyperbolic form corresponding to real $s$. However, the complete E. I. III between the limits $0 < \phi < \frac{1}{2}\pi$, or $0 < u < K$, $0 < e < 1$, can always be expressed by the E. I. I and II, as Legendre pointed out.

**11.6.** The standard forms are given above to which an elliptic integral must be reduced when the result is required in a numerical form taken from the Tables. But in a practical problem the integral arises in a general algebraical form, and theory shows that the result can always be made, by a suitable substitution, to depend on three differential elements, of the I, II, III kind,

$$\text{I} \quad \frac{ds}{\sqrt{S}}$$

$$\text{II} \quad (s-a)\frac{ds}{\sqrt{S}}$$

$$\text{III} \quad \frac{1}{(s-\sigma)}\frac{ds}{\sqrt{S}}$$

where $S$ is a cubic in the variable $s$ which may be written, when resolved into three factors,

$$S = 4 \cdot s - s_1 \cdot s - s_2 \cdot s - s_3$$

in the sequence $\infty > s_1 > s_2 > s_3 > -\infty$, and normalised to a standard form of zero degree these differential elements are

$$\text{I} \quad \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}}$$

$$\text{II} \quad \frac{s-a}{\sqrt{s_1 - s_3}}\frac{ds}{\sqrt{S}}$$

$$\text{III} \quad \frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}}$$

$\Sigma$ denoting the value of $S$ when $s = \sigma$.

The relative positions of $s$ and $\sigma$ in the intervals of the sequence require preliminary consideration before introducing the Elliptic Functions and their notation.

**11.7.** For the E. I. I and its representation in a tabular form with

$$\kappa^2 = \frac{s_2 - s_3}{s_1 - s_3}, \qquad \kappa'^2 = \frac{s_1 - s_2}{s_1 - s_3},$$

$$K = \int_{s_1,\, s_3}^{\infty,\, s_2} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}}, \qquad K' = \int_{s_2,\, -\infty}^{s_1,\, s_3} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}},$$

and utilizing the inverse notation, then in the first interval of the sequence,

$$\infty > s > s_1$$

$$eK = \int_s^{\infty} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3}{s - s_3}} = \mathrm{cn}^{-1}\sqrt{\frac{s - s_1}{s - s_3}} = \mathrm{dn}^{-1}\sqrt{\frac{s - s_2}{s - s_3}}$$

$$(1 - e)K = \int_{s_1}^{s} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s - s_1}{s - s_2}} = \mathrm{cn}^{-1}\sqrt{\frac{s_1 - s_2}{s - s_2}} = \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s_2 \cdot s - s_3}{s_1 - s_3 \cdot s - s_2}}$$

indicating the substitutions,

$$\frac{s_1 - s_3}{s - s_3} = \sin^2\phi = \mathrm{sn}^2\, eK, \qquad \frac{s - s_1}{s - s_2} = \sin^2\psi = \mathrm{sn}^2\,(1 - e)K.$$

In the next interval $S$ is negative, and the comodulus $\kappa'$ is required.

$$s_1 > s > s_2$$

$$fK' = \int^{s_1} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s}{s_1 - s_2}} = \mathrm{cn}^{-1}\sqrt{\frac{s - s_2}{s_1 - s_2}} = \mathrm{dn}^{-1}\sqrt{\frac{s - s_3}{s_1 - s_3}}$$

$$(1 - f)K' = \int_{s_2}^{s} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3 \cdot s - s_2}{s_1 - s_2 \cdot s - s_3}} = \mathrm{cn}^{-1}\sqrt{\frac{s_2 - s_3 \cdot s_1 - s}{s_1 - s_2 \cdot s - s_1}}$$

$$= \mathrm{dn}^{-1}\sqrt{\frac{s_2 - s_3}{s - s_3}}$$

$S$ is positive again in the next interval, and the modulus is $\kappa$.

$$s_2 > s > s_3$$

$$(1 - e)K = \int_s^{s_2} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3 \cdot s_2 - s}{s_2 - s_3 \cdot s_1 - s}} = \mathrm{cn}^{-1}\sqrt{\frac{s_1 - s_2 \cdot s - s_3}{s_2 - s_3 \cdot s_1 - s}}$$

$$= \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s_2}{s_1 - s}}$$

$$eK = \int_{s_3}^{s} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s - s_3}{s_2 - s_3}} = \mathrm{cn}^{-1}\sqrt{\frac{s_2 - s}{s_2 - s_3}} = \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s}{s_1 - s_3}}$$

indicating the substitutions,

$$\frac{s_1 - s_2}{s_1 - s} = \Delta^2\psi = \mathrm{dn}^2\,(1 - e)K, \qquad \frac{s - s_3}{s_2 - s_3} = \sin^2\phi = \mathrm{sn}^2\, eK$$

$$s = s_2 \sin^2\phi + s_3 \cos^2\phi.$$

$S$ is negative again in the last interval, and the modulus $\kappa'$.

$$s_3 > s > -\infty$$

$$(1-f)K' = \int_s^{s_3} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_3 - s}{s_2 - s}} = \mathrm{cn}^{-1}\sqrt{\frac{s_2 - s_3}{s_2 - s}} = \mathrm{dn}^{-1}\sqrt{\frac{s_2 - s_3 \cdot s_1 - s}{s_1 - s_3 \cdot s_2 - s}}$$

$$fK' = \int_{-\infty}^{s} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3}{s_1 - s}} = \mathrm{cn}^{-1}\sqrt{\frac{s_3 - s}{s_1 - s}} = \mathrm{dn}^{-1}\sqrt{\frac{s_2 - s}{s_1 - s}}$$

**11.8.** For the notation of the E. I. II and the various reductions, take the treatment given in the Trans. Am. Math. Soc., 1907, vol. 8, p. 450. The Jacobian Zeta Function and the *Er, Gr* of the Tables, are defined by the standard integral

$$\int_{s_3}^{s} \frac{s_1 - s}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \int_0^{\phi} \Delta\phi \cdot d\phi = E\phi = \int_0^{e} \mathrm{dn}^2(eK) \cdot d(eK) = E\,\mathrm{am}\, eK = eH + \mathrm{zn}\, eK,$$

or,

$$\int_{s_2}^{\sigma} \frac{\sigma - s_3}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int_0^{f} \mathrm{dn}^2(fK') \cdot d(fK') = E\,\mathrm{am}\, fK' = fH' + \mathrm{zn}\, fK',$$

where zn is Jacobi's Zeta Function, and $H, H'$ the complete E. I. II to modulus $\kappa$, $\kappa'$, defined by,

$$H = \int_0^{\frac{\pi}{2}} \Delta(\phi, \kappa)\, d\phi = \int_0^{1} \mathrm{dn}^2(eK) \cdot d(eK)$$

$$H' = \int_0^{\frac{\pi}{2}} \Delta(\phi, \kappa')\, d\phi = \int_0^{1} \mathrm{dn}^2(fK') \cdot d(fK').$$

The function zn $u$ is derived by logarithmic differentiation of $\Theta u$,

$\mathrm{zn}\, u = \dfrac{d \log \Theta u}{du}$, or concisely,

$$\Theta u = \exp. \int \mathrm{zn}\, u \cdot du,$$

and a function zs $u$ is derived similarly from

$$\mathrm{zs}\, u = \frac{d \log Hu}{du}$$

$$= \frac{d \log \Theta u}{du} + \frac{d \log \mathrm{sn}\, u}{du}$$

$$= \mathrm{zn}\, u + \frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}.$$

For the incomplete E. I. II in the regions,

$$\infty > s > s_1 > s_2 > s > s_3$$

and

$$\mathrm{sn}^2\, eK = \frac{s_1 - s_3}{s - s_3} \text{ or } \frac{s - s_3}{s_2 - s_3},$$

$$\int_s^{s_1} \frac{s-s_1}{\sqrt{s_1-s_3}} \frac{ds}{\sqrt{S}} = \int_s^{s_2} \frac{s_2-s}{s-s_3} \frac{\sqrt{s-s_3}}{\sqrt{S}}\, ds = -(1-e)H + \operatorname{zs} eK$$

$$\int \frac{s-s_2}{\sqrt{s_1-s_3}} \frac{ds}{\sqrt{S}} = \kappa^2 \int \frac{s_1-s}{s-s_3} \frac{\sqrt{s_1-s_3}}{\sqrt{S}}\, ds = -(1-e)(H-\kappa'^2 K) + \operatorname{zs} eK$$

$$\int \frac{s-s_3}{\sqrt{s_1-s_3}} \frac{ds}{\sqrt{S}} = \int \frac{s_2-s_3}{s-s_3} \frac{\sqrt{s_1-s_3}}{\sqrt{S}}\, ds = (1-e)(K-H) + \operatorname{zs} eK$$

the integrals being $\infty$ at the upper limit, $s = \infty$, or at the lower limit, $s = s_3$ where $e = 0$ and $\operatorname{zs} eK = \infty$.

So also,

$$\int_{s,\, s_1}^{\infty,\, s} \frac{s-s_2}{s-s_3} \frac{\sqrt{s_1-s_3}}{\sqrt{S}}\, ds = \int_{s_3,\, s}^{s,\, s_2} \frac{s_1-s}{\sqrt{s_1-s_3}} \frac{ds}{\sqrt{S}} = \begin{matrix} eH + \operatorname{zn} eK \\ (1-e)H - \operatorname{zn} eK \end{matrix}$$

$$\int \frac{s-s_1}{s-s_3} \frac{\sqrt{s_1-s_3}}{\sqrt{S}}\, ds = \int \frac{s_2-s}{\sqrt{s_1-s_3}} \frac{ds}{\sqrt{S}} = \begin{matrix} e(H-\kappa'^2 K) + \operatorname{zn} eK \\ (1-e)(H-\kappa'^2 K) - \operatorname{zn} eK \end{matrix}$$

$$\int \frac{s_2-s_3}{s-s_3} \frac{\sqrt{s_1-s_3}}{\sqrt{S}}\, ds = \int \frac{s-s_3}{\sqrt{s_1-s_3}} \frac{ds}{\sqrt{S}} = \begin{matrix} e(K-H) - \operatorname{zn} eK \\ (1-e)(K-H) + \operatorname{zn} eK \end{matrix}$$

Similarly, for the variable $\sigma$ in the regions

$$s_1 > \sigma > s_2 > s_3 > \sigma > -\infty$$

$\Sigma$ negative, and

$$\operatorname{sn}^2 fK' = \frac{s_1-\sigma}{s_1-s_2} \text{ or } \frac{s_1-s_3}{s_1-\sigma}$$

$$\int_{\sigma,\, s_2}^{s_1,\, \sigma} \frac{s_1-\sigma}{\sqrt{s_1-s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int_{-\infty,\, \sigma}^{\sigma,\, s_3} \frac{s_1-s_2}{s_1-\sigma} \frac{\sqrt{s_1-s_3}}{\sqrt{-\Sigma}}\, d\sigma = \begin{matrix} f(K'-H') - \operatorname{zn} fK' \\ (1-f)(K'-H') + \operatorname{zn} fK' \end{matrix}$$

$$\int \frac{\sigma-s_2}{\sqrt{s_1-s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int \frac{s_3-\sigma}{s_1-\sigma} \frac{\sqrt{s_1-s_3}}{\sqrt{-\Sigma}}\, d\sigma = \begin{matrix} f(H'-\kappa'^2 K') + \operatorname{zn} fK' \\ (1-f)(H'-\kappa'^2 K') - \operatorname{zn} fK' \end{matrix}$$

$$\int \frac{\sigma-s_3}{\sqrt{s_1-s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int \frac{s_2-\sigma}{s_1-\sigma} \frac{\sqrt{s_1-s_3}}{\sqrt{-\Sigma}}\, d\sigma = \begin{matrix} fH' + \operatorname{zn} fK' \\ (1-f)H' - \operatorname{zn} fK' \end{matrix}$$

$$\int_{s_2}^{\sigma} \frac{s_1-s_2}{s_1-\sigma} \frac{\sqrt{s_1-s_3}}{\sqrt{-\Sigma}}\, d\sigma = \int_{\sigma}^{s_3} \frac{s_1-\sigma}{\sqrt{s_1-s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = (1-f)(K'-H') + \operatorname{zs} fK'$$

$$\kappa'^2 \int \frac{s_3-\sigma}{s_1-\sigma} \frac{\sqrt{s_1-s_3}}{\sqrt{-\Sigma}}\, d\sigma = \int \frac{s_2-\sigma}{\sqrt{s_1-s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = -(1-f)(H'-\kappa^2 K') + \operatorname{zs} fK'$$

$$\int \frac{s_2-\sigma}{s_1-\sigma} \frac{\sqrt{s_1-s_3}}{\sqrt{-\Sigma}}\, d\sigma = \int \frac{s_3-\sigma}{\sqrt{s_1-s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = -(1-f)H' + \operatorname{zs} fK'$$

these last three integrals being infinite at the upper limit, $\sigma = s_1$, or lower limit $\sigma = -\infty$, where $f = 0$, $\operatorname{zs} fK' = \infty$.

Putting $e = 1$ or $f = 1$ any of these forms will give the complete E. I. II, noticing that zn $K'$ and zs $K'$ are zero.

**11.9.** In dealing practically with an E. I. III it is advisable to study it first in the algebraical form of Weierstrass,

$$\int \frac{\frac{1}{2}\sqrt{\Sigma}\, ds}{(s-\sigma)\sqrt{S}},$$

where $S = 4\cdot s - s_1\cdot s - s_2\cdot s - s_3$, $\Sigma$ the same function of $\sigma$, and begin by examining the sequence of the quantities $s$, $\sigma$, $s_1$, $s_2$, $s_3$

Then in the region

$$s > s_1 > s_2 > \sigma > s_3,$$

put

$$s - s_3 = \frac{s_1 - s_3}{\operatorname{sn}^2 u},\quad \sigma - s_3 = (s_2 - s_3)\operatorname{sn}^2 v,\quad \kappa^2 = \frac{s_2 - s_3}{s_1 - s_3},$$

$$s - \sigma = \frac{s_1 - s_3}{\operatorname{sn}^2 u}(1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v),\quad \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = du,$$

$$\sqrt{\Sigma} = \sqrt{s_1 - s_3}\,(s_2 - s_3)\operatorname{sn} v \operatorname{cn} v \operatorname{dn} v, \text{ making}$$

$$\int \frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = \int \frac{\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}\, du = \Pi(u, v).$$

But in the region,

$$\sigma > s_1 > s_2 > s > s_3,$$

$$s - s_3 = (s_2 - s_3)\operatorname{sn}^2 u,\quad \sigma - s_3 = \frac{s_1 - s_3}{\operatorname{sn}^2 v},\quad \frac{1}{2}\sqrt{\Sigma} = (s_1 - s_3)^{\frac{3}{2}}\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn}^3 v},$$

$$\sigma - s = \frac{s_1 - s_3}{\operatorname{sn}^2 v}(1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v),$$

making,

$$\int \frac{\frac{1}{2}\sqrt{\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}} = \int \frac{\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}\, du}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} = \Pi_1 = \Pi(u, v) + u\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}.$$

In a dynamical application the sequence is usually

$$s > s_1 > \sigma > s_2 > s > s_3$$

or

$$s > s_1 > s_2 > s > s_3 > \sigma,$$

making $\Sigma$ negative, and the E. I. III is then called circular; the parameter $v$ is then imaginary, and the expression by the Theta function is illusory.

The complete E. I. III, however, was shown by Legendre to be tractable and falls into four classes, lettered $(l')$ $(m')$, p. 138, $(i')$, $(k')$, pp. 133, 134 (Fonctions elliptiques, I).

$s_1 > \sigma > s_2$

$$\operatorname{sn}^2 fK' = \frac{s_1 - \sigma}{s_1 - s_2}$$

$$\operatorname{cn}^2 fK' = \frac{\sigma - s_2}{s_1 - s_2}$$

$$\operatorname{dn}^2 fK' = \frac{\sigma - s_3}{s_1 - s_3}$$

A.
$$\infty > s > s_1 \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = A(fK') = \tfrac{1}{2}\pi(1-f) - K\,\mathrm{zn}\,fK'$$

B.
$$s_2 > s > s_3 \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}} = B(fK') = \tfrac{1}{2}\pi f + K\,\mathrm{zn}\,fK'$$

$$A + B = \tfrac{1}{2}\pi.$$

$s_3 > \sigma > -\infty$

$$\mathrm{sn}^2\, fK' = \frac{s_1 - s_3}{s_1 - \sigma}$$

$$\mathrm{cn}^2\, fK' = \frac{s_3 - \sigma}{s_1 - \sigma}$$

$$\mathrm{dn}^2\, fK' = \frac{s_2 - \sigma}{s_1 - \sigma}$$

C.
$$\infty > s > s_1 \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = C(fK') = K\,\mathrm{zs}\,fK' - \tfrac{1}{2}\pi(1-f)$$

D.
$$s_2 > s > s_3 \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = D(fK') = K\,\mathrm{zs}\,fK' + \tfrac{1}{2}\pi f$$

$$D - C = \tfrac{1}{2}\pi.$$

# TABLES OF ELLIPTIC FUNCTIONS

By Col. R. L. Hippisley

K = 1.5737921309, K′ = 3.831742000, E = 1.5678090740, E′ = 1.012663506,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01748 65792 | 1 0 | 0.00006 64649 | 1.00000 05812 | 0.01745 23906 |
| 2 | 0.03497 31585 | 2 0 | 0.00013 28485 | 1.00000 23240 | 0.03489 94650 |
| 3 | 0.05245 97377 | 3 0 | 0.00019 90699 | 1.00000 52264 | 0.05233 59088 |
| 4 | 0.06994 63169 | 4 0 | 0.00026 50480 | 1.00000 92847 | 0.06975 64107 |
| 5 | 0.08743 28962 | 5 1 | 0.00033 07023 | 1.00001 44942 | 0.08715 56642 |
| 6 | 0.10491 94754 | 6 1 | 0.00039 59525 | 1.00002 08483 | 0.10452 83693 |
| 7 | 0.12240 60546 | 7 1 | 0.00046 07190 | 1.00002 83393 | 0.12186 92343 |
| 8 | 0.13989 26338 | 8 1 | 0.00052 49226 | 1.00003 69582 | 0.13917 29770 |
| 9 | 0.15737 92131 | 9 1 | 0.00058 84849 | 1.00004 66945 | 0.15643 43264 |
| 10 | 0.17486 57923 | 10 1 | 0.00065 13283 | 1.00005 75362 | 0.17364 80247 |
| 11 | 0.19235 23716 | 11 1 | 0.00071 33760 | 1.00006 94702 | 0.19080 88283 |
| 12 | 0.20983 89508 | 12 1 | 0.00077 45523 | 1.00008 24819 | 0.20791 15101 |
| 13 | 0.22732 55300 | 13 1 | 0.00083 47824 | 1.00009 65555 | 0.22495 08603 |
| 14 | 0.24481 21092 | 14 2 | 0.00089 39929 | 1.00011 16738 | 0.24192 16887 |
| 15 | 0.26229 86885 | 15 2 | 0.00095 21114 | 1.00012 78184 | 0.25881 88257 |
| 16 | 0.27978 52677 | 16 2 | 0.00100 90670 | 1.00014 49696 | 0.27563 71244 |
| 17 | 0.29727 18469 | 17 2 | 0.00106 47903 | 1.00016 31066 | 0.29237 14618 |
| 18 | 0.31475 84262 | 18 2 | 0.00111 92132 | 1.00018 22072 | 0.30901 67404 |
| 19 | 0.33224 50054 | 19 2 | 0.00117 22694 | 1.00020 22482 | 0.32556 78900 |
| 20 | 0.34973 15846 | 20 2 | 0.00122 38941 | 1.00022 32051 | 0.34201 98690 |
| 21 | 0.36721 81639 | 21 2 | 0.00127 40244 | 1.00024 50525 | 0.35836 76658 |
| 22 | 0.38470 47431 | 22 2 | 0.00132 25992 | 1.00026 77636 | 0.37460 63009 |
| 23 | 0.40219 13223 | 23 2 | 0.00136 95594 | 1.00029 13109 | 0.39073 08277 |
| 24 | 0.41967 79016 | 24 2 | 0.00141 48476 | 1.00031 56657 | 0.40673 63347 |
| 25 | 0.43716 44808 | 25 3 | 0.00145 84087 | 1.00034 07982 | 0.42261 79464 |
| 26 | 0.45465 10600 | 26 3 | 0.00150 01897 | 1.00036 66779 | 0.43837 08251 |
| 27 | 0.47213 76393 | 27 3 | 0.00154 01398 | 1.00039 32731 | 0.45399 01723 |
| 28 | 0.48962 42185 | 28 3 | 0.00157 82103 | 1.00042 05516 | 0.46947 12303 |
| 29 | 0.50711 07977 | 29 3 | 0.00161 43549 | 1.00044 84801 | 0.48480 92833 |
| 30 | 0.52459 73770 | 30 3 | 0.00164 85297 | 1.00047 70246 | 0.49999 96593 |
| 31 | 0.54208 39562 | 31 3 | 0.00168 06931 | 1.00050 61502 | 0.51503 77311 |
| 32 | 0.55957 05354 | 32 3 | 0.00171 08062 | 1.00053 58215 | 0.52991 89180 |
| 33 | 0.57705 71147 | 33 3 | 0.00173 88322 | 1.00056 60024 | 0.54463 86870 |
| 34 | 0.59454 36939 | 34 3 | 0.00176 47373 | 1.00059 66561 | 0.55919 25543 |
| 35 | 0.61203 02731 | 35 3 | 0.00178 84901 | 1.00062 77451 | 0.57357 60867 |
| 36 | 0.62951 68524 | 36 3 | 0.00181 00617 | 1.00065 92318 | 0.58778 49028 |
| 37 | 0.64700 34316 | 37 3 | 0.00182 94261 | 1.00069 10776 | 0.60181 46744 |
| 38 | 0.66449 00108 | 38 3 | 0.00184 65599 | 1.00072 32438 | 0.61566 11280 |
| 39 | 0.68197 65900 | 39 3 | 0.00186 14423 | 1.00075 56912 | 0.62932 00458 |
| 40 | 0.69946 31693 | 40 3 | 0.00187 40556 | 1.00078 83803 | 0.64278 72670 |
| 41 | 0.71694 97485 | 41 4 | 0.00188 43845 | 1.00082 12712 | 0.65605 86895 |
| 42 | 0.73443 63278 | 42 4 | 0.00189 24166 | 1.00085 43239 | 0.66913 02706 |
| 43 | 0.75192 29070 | 43 4 | 0.00189 81424 | 1.00088 74981 | 0.68199 80287 |
| 44 | 0.76940 94862 | 44 4 | 0.00190 15552 | 1.00092 07533 | 0.69465 80439 |
| 45 | 78689 60655 | 45 4 | 0.00190 26510 | 1.00095 40492 | 0.70710 64600 |
| 90° r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 5°$

$q = 0.000476569916867$, $\Theta 0 = 0.9990468602$, $H(K) = 0.2955029021$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90° r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.00190 80984 | 0.00000 00000 | 90° 0′ | 1.57379 21309 | 90 |
| 0.99984 76949 | 1.00190 75172 | 0.00006 63384 | 89 0 | 1.55630 55517 | 89 |
| 0.99939 08259 | 1.00190 57743 | 0.00013 25961 | 88 0 | 1.53881 89724 | 88 |
| 0.99862 95323 | 1.00190 28720 | 0.00019 86928 | 87 0 | 1.52133 23932 | 87 |
| 0.99756 40458 | 1.00189 88136 | 0.00026 45481 | 86 0 | 1.50384 58140 | 86 |
| 0.99619 46912 | 1.00189 36042 | 0.00033 00820 | 85 1 | 1.48635 92347 | 85 |
| 0.99452 18855 | 1.00188 72501 | 0.00039 52149 | 84 1 | 1.46887 26555 | 84 |
| 0.99254 61382 | 1.00187 97590 | 0.00045 98676 | 83 1 | 1.45138 60763 | 83 |
| 0.99026 80513 | 1.00187 11401 | 0.00052 39616 | 82 1 | 1.43389 94971 | 82 |
| 0.98768 83186 | 1.00186 14039 | 0.00058 74190 | 81 1 | 1.41641 29178 | 81 |
| 0.98480 77260 | 1.00185 05621 | 0.00065 01626 | 80 1 | 1.39892 63386 | 80 |
| 0.98162 71510 | 1.00183 86282 | 0.00071 21163 | 79 1 | 1.38143 97593 | 79 |
| 0.97814 75623 | 1.00182 56165 | 0.00077 32046 | 78 1 | 1.36395 31801 | 78 |
| 0.97437 00200 | 1.00181 15429 | 0.00083 33534 | 77 1 | 1.34646 66009 | 77 |
| 0.97029 56747 | 1.00179 64246 | 0.00089 24894 | 76 2 | 1.32898 00217 | 76 |
| 0.96592 57675 | 1.00178 02800 | 0.00095 05409 | 75 2 | 1.31149 34424 | 75 |
| 0.96126 16296 | 1.00176 31288 | 0.00100 74371 | 74 2 | 1.29400 68632 | 74 |
| 0.95630 46817 | 1.00174 49918 | 0.00106 31089 | 73 2 | 1.27652 02840 | 73 |
| 0.95105 64338 | 1.00172 58912 | 0.00111 74885 | 72 2 | 1.25903 37047 | 72 |
| 0.94551 84846 | 1.00170 58502 | 0.00117 05097 | 71 2 | 1.24154 71255 | 71 |
| 0.93969 25209 | 1.00168 48932 | 0.00122 21081 | 70 2 | 1.22406 05463 | 70 |
| 0.93358 03176 | 1.00166 30459 | 0.00127 22208 | 69 2 | 1.20657 39670 | 69 |
| 0.92718 37364 | 1.00164 03347 | 0.00132 07868 | 68 2 | 1.18908 73878 | 68 |
| 0.92050 47258 | 1.00161 67874 | 0.00136 77470 | 67 2 | 1.17160 08086 | 67 |
| 0.91354 53203 | 1.00159 24327 | 0.00141 30440 | 66 3 | 1.15411 42293 | 66 |
| 0.90630 76400 | 1.00156 73002 | 0.00145 66228 | 65 3 | 1.13662 76501 | 65 |
| 0.89879 38894 | 1.00154 14205 | 0.00149 84301 | 64 3 | 1.11914 10709 | 64 |
| 0.89100 63574 | 1.00151 48252 | 0.00153 84151 | 63 3 | 1.10165 44916 | 63 |
| 0.88294 74161 | 1.00148 75467 | 0.00157 65289 | 62 3 | 1.08416 79124 | 62 |
| 0.87461 95204 | 1.00145 96182 | 0.00161 27250 | 61 3 | 1.06668 13332 | 61 |
| 0.86602 52071 | 1.00143 10738 | 0.00164 69592 | 60 3 | 1.04919 47539 | 60 |
| 0.85716 70941 | 1.00140 19481 | 0.00167 91897 | 59 3 | 1.03170 81747 | 59 |
| 0.84804 78798 | 1.00137 22768 | 0.00170 93771 | 58 3 | 1.01422 15955 | 58 |
| 0.83867 03419 | 1.00134 20959 | 0.00173 74846 | 57 3 | 0.99673 50162 | 57 |
| 0.82903 73370 | 1.00131 14423 | 0.00176 34776 | 56 3 | 0.97924 84370 | 56 |
| 0.81915 17995 | 1.00128 03532 | 0.00178 73244 | 55 3 | 0.96176 18578 | 55 |
| 0.80901 67404 | 1.00124 88666 | 0.00180 89958 | 54 3 | 0.94427 52785 | 54 |
| 0.79863 52473 | 1.00121 70208 | 0.00182 84651 | 53 3 | 0.92678 86993 | 53 |
| 0.78801 04823 | 1.00118 48546 | 0.00184 57085 | 52 3 | 0.90930 21201 | 52 |
| 0.77714 56818 | 1.00115 24072 | 0.00186 07047 | 51 3 | 0.89181 55409 | 51 |
| 0.76604 41556 | 1.00111 97181 | 0.00187 34353 | 50 3 | 0.87432 89616 | 50 |
| 0.75470 92851 | 1.00108 68272 | 0.00188 38846 | 49 3 | 0.85684 23824 | 49 |
| 0.74314 45232 | 1.00105 37745 | 0.00189 20395 | 48 3 | 0.83935 58031 | 48 |
| 0.73135 33926 | 1.00102 06003 | 0.00189 78900 | 47 3 | 0.82186 92239 | 47 |
| 0.71933 94850 | 1.00098 73450 | 0.00190 14287 | 46 4 | 0.80438 26447 | 46 |
| 0.70710 64600 | 1.00095 40492 | 0.00190 26510 | 45 4 | 0.78689 60655 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 1.5828428043, K′ = 3.153385252, E = 1.5588871966, E′ = 1.040114396,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01758 71423 | 1 0 | 0.00026 61187 | 1.00000 23404 | 0.01745 21509 |
| 2 | 0.03517 42845 | 2 1 | 0.00053 19095 | 1.00000 93587 | 0.03489 89861 |
| 3 | 0.05276 14268 | 3 1 | 0.00079 70448 | 1.00002 10463 | 0.05233 51918 |
| 4 | 0.07034 85691 | 4 2 | 0.00106 11979 | 1.00003 73890 | 0.06975 54570 |
| 5 | 0.08793 57113 | 5 2 | 0.00132 40433 | 1.00005 83670 | 0.08715 44758 |
| 6 | 0 10552 28536 | 6 3 | 0.00158 52573 | 1.00008 39546 | 0.10452 69489 |
| 7 | 0.12310 99959 | 7 3 | 0.00184 45182 | 1.00011 41206 | 0.12186 75849 |
| 8 | 0.14069 71382 | 8 4 | 0.00210 15066 | 1.00014 88284 | 0.13917 11019 |
| 9 | 0.15828 42804 | 9 4 | 0.00235 59064 | 1.00018 80356 | 0.15643 22298 |
| 10 | 0.17587 14227 | 10 5 | 0.00260 74044 | 1.00023 16945 | 0.17364 57109 |
| 11 | 0.19345 85650 | 11 5 | 0.00285 56913 | 1.00027 97518 | 0.19080 63023 |
| 12 | 0.21104 57072 | 12 5 | 0.00310 04619 | 1.00033 21491 | 0.20790 87771 |
| 13 | 0.22863 28495 | 13 6 | 0.00334 14153 | 1.00038 88224 | 0.22494 79261 |
| 14 | 0.24621 99918 | 14 6 | 0.00357 82555 | 1.00044 97028 | 0.24191 85595 |
| 15 | 0.26380 71340 | 15 7 | 0.00381 06920 | 1.00051 47160 | 0.25881 55080 |
| 16 | 0.28139 42763 | 16 7 | 0.00403 84394 | 1.00058 37829 | 0.27563 36252 |
| 17 | 0.29898 14186 | 17 7 | 0.00426 12186 | 1.00065 68193 | 0.29236 77883 |
| 18 | 0.31656 85609 | 18 8 | 0.00447 87567 | 1.00073 37362 | 0.30901 29003 |
| 19 | 0.33415 57031 | 19 8 | 0.00469 07873 | 1.00081 44399 | 0.32556 38912 |
| 20 | 0.35174 28454 | 20 8 | 0.00489 70511 | 1.00089 88322 | 0.34201 57197 |
| 21 | 0.36932 99877 | 21 9 | 0.00509 72961 | 1.00098 68100 | 0.35836 33745 |
| 22 | 0.38691 71299 | 22 9 | 0.00529 12778 | 1.00107 82664 | 0.37460 18764 |
| 23 | 0.40450 42722 | 23 9 | 0.00547 87596 | 1.00117 30898 | 0.39072 62791 |
| 24 | 0.42209 14145 | 24 10 | 0.00565 95131 | 1.00127 11647 | 0.40673 16711 |
| 25 | 0.43967 85568 | 25 10 | 0.00583 33185 | 1.00137 23717 | 0.42261 31771 |
| 26 | 0.45726 56990 | 26 10 | 0.00599 99643 | 1.00147 65874 | 0.43836 59597 |
| 27 | 0.47485 28413 | 27 11 | 0.00615 92485 | 1.00158 36848 | 0.45398 52206 |
| 28 | 0.49243 99836 | 28 11 | 0.00631 09780 | 1.00169 35336 | 0.46946 62019 |
| 29 | 0.51002 71258 | 29 11 | 0.00645 49693 | 1.00180 59998 | 0.48480 41881 |
| 30 | 0.52761 42681 | 30 11 | 0.00659 10484 | 1.00192 09464 | 0.49999 45073 |
| 31 | 0.54520 14104 | 31 12 | 0.00671 90513 | 1.00203 82334 | 0.51503 25321 |
| 32 | 0.56278 85526 | 32 12 | 0.00683 88242 | 1.00215 77178 | 0.52991 36820 |
| 33 | 0.58037 56949 | 33 12 | 0.00695 02232 | 1.00227 92542 | 0.54463 34239 |
| 34 | 0.59796 28372 | 34 12 | 0.00705 31150 | 1.00240 26944 | 0.55918 72740 |
| 35 | 0.61554 99795 | 35 12 | 0.00714 73769 | 1.00252 78880 | 0.57357 07990 |
| 36 | 0.63313 71217 | 36 13 | 0.00723 28968 | 1.00265 46826 | 0.58777 96173 |
| 37 | 0.65072 42640 | 37 13 | 0.00730 95735 | 1.00278 29236 | 0.60180 94008 |
| 38 | 0.66831 14063 | 38 13 | 0.00737 73166 | 1.00291 24548 | 0.61565 58756 |
| 39 | 0.68589 85485 | 39 13 | 0.00743 60469 | 1.00304 31183 | 0.62931 48239 |
| 40 | 0.70348 56908 | 40 13 | 0.00748 56962 | 1.00317 47551 | 0.64278 20847 |
| 41 | 0.72107 28331 | 41 13 | 0.00752 62073 | 1.00330 72046 | 0.65605 35555 |
| 42 | 0.73865 99754 | 42 13 | 0.00755 75345 | 1.00344 03056 | 0.66912 51936 |
| 43 | 0.75624 71176 | 43 13 | 0.00757 96433 | 1.00357 38959 | 0.68199 30169 |
| 44 | 0.77383 42599 | 44 13 | 0.00759 25102 | 1.00370 78127 | 0.69465 31055 |
| 45 | 0.79142 14022 | 45 13 | 0.00759 61235 | 1.00384 18928 | 0.70710 16026 |
| 90-r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 10°$

$q = 0.00191359459017$, $\Theta 0 = 0.9961728108$, $HK = 0.418305976553$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.00768 37857 | 0.00000 00000 | 90° 0′ | 1.58284 28043 | 90 |
| 0.99984 76907 | 1.00768 14453 | 0.00026 40908 | 89 0 | 1.56525 56621 | 89 |
| 0.99939 08092 | 1.00767 44270 | 0.00052 78635 | 88 1 | 1.54766 85198 | 88 |
| 0.99862 94947 | 1.00766 27394 | 0.00079 10004 | 87 1 | 1.53008 13775 | 87 |
| 0.99756 39792 | 1.00764 63966 | 0.00105 31846 | 86 2 | 1.51249 42353 | 86 |
| 0.99619 45873 | 1.00762 54187 | 0.00131 41001 | 85 2 | 1.49490 70930 | 85 |
| 0.99452 17362 | 1.00759 98311 | 0.00157 34327 | 84 3 | 1.47731 99507 | 84 |
| 0.99254 59357 | 1.00756 96650 | 0.00183 08697 | 83 3 | 1.45973 28084 | 83 |
| 0.99026 77878 | 1.00753 49572 | 0.00208 61008 | 82 4 | 1.44214 56662 | 82 |
| 0.98768 79866 | 1.00749 57500 | 0.00233 88183 | 81 4 | 1.42445 85239 | 81 |
| 0.98480 73181 | 1.00745 20912 | 0.00258 87173 | 80 4 | 1.40697 13816 | 80 |
| 0.98162 66600 | 1.00740 40338 | 0.00283 54962 | 79 5 | 1.38938 42394 | 79 |
| 0.97814 69814 | 1.00735 16366 | 0.00307 88572 | 78 5 | 1.37179 70971 | 78 |
| 0.97436 93426 | 1.00729 49632 | 0.00331 85063 | 77 6 | 1.35420 99548 | 77 |
| 0.97029 48945 | 1.00723 40828 | 0.00355 41538 | 76 6 | 1.33662 28125 | 76 |
| 0.96592 48785 | 1.00716 90696 | 0.00378 55150 | 75 7 | 1.31903 56703 | 75 |
| 0.96126 06262 | 1.00710 00027 | 0.00401 23098 | 74 7 | 1.30144 85280 | 74 |
| 0.95630 35586 | 1.00702 69663 | 0.00423 42636 | 73 7 | 1.28386 13857 | 73 |
| 0.95105 51861 | 1.00695 00494 | 0.00445 11077 | 72 8 | 1.26627 42435 | 72 |
| 0.94551 71076 | 1.00686 93457 | 0.00466 25790 | 71 8 | 1.24868 71012 | 71 |
| 0.93969 10107 | 1.00678 49535 | 0.00486 84209 | 70 8 | 1.23109 99589 | 70 |
| 0.93357 86703 | 1.00669 69756 | 0.00506 83836 | 69 9 | 1.21351 28167 | 69 |
| 0.92718 19488 | 1.00660 55192 | 0.00526 22237 | 68 9 | 1.19592 56744 | 68 |
| 0.92050 27950 | 1.00651 06958 | 0.00544 97055 | 67 9 | 1.17833 85321 | 67 |
| 0.91354 32440 | 1.00641 26209 | 0.00563 06006 | 66 10 | 1.16075 13898 | 66 |
| 0.90630 54160 | 1.00631 14139 | 0.00580 46884 | 65 10 | 1.14316 42476 | 65 |
| 0.89879 15164 | 1.00620 71982 | 0.00597 17561 | 64 10 | 1.12557 71053 | 64 |
| 0.89100 38343 | 1.00610 01007 | 0.00613 15997 | 63 11 | 1.10798 99630 | 63 |
| 0.88294 47424 | 1.00599 02520 | 0.00628 40232 | 62 11 | 1.09040 28208 | 62 |
| 0.87461 66961 | 1.00587 77858 | 0.00642 88398 | 61 11 | 1.07281 56785 | 61 |
| 0.86602 22325 | 1.00576 28392 | 0.00656 58716 | 60 12 | 1.05522 85362 | 60 |
| 0.85716 39703 | 1.00564 55522 | 0.00669 49498 | 59 12 | 1.03764 13940 | 59 |
| 0.84804 46080 | 1.00552 60678 | 0.00681 59154 | 58 12 | 1.02005 42517 | 58 |
| 0.83866 69240 | 1.00540 45314 | 0.00692 86187 | 57 12 | 1.00246 71094 | 57 |
| 0.82903 37754 | 1.00528 10912 | 0.00703 29201 | 56 12 | 0.98487 99671 | 56 |
| 0.81914 80969 | 1.00515 58975 | 0.00712 86900 | 55 12 | 0.96729 28249 | 55 |
| 0.80901 29003 | 1.00502 91030 | 0.00721 58089 | 54 13 | 0.94970 56826 | 54 |
| 0.79863 12733 | 1.00490 08620 | 0.00729 41679 | 53 13 | 0.93211 85403 | 53 |
| 0.78800 63786 | 1.00477 13308 | 0.00736 36683 | 52 13 | 0.91453 13981 | 52 |
| 0.77714 14532 | 1.00464 06672 | 0.00742 42224 | 51 13 | 0.89694 42558 | 51 |
| 0.76603 98071 | 1.00450 90305 | 0.00747 57531 | 50 13 | 0.87935 71135 | 50 |
| 0.75470 48222 | 1.00437 65809 | 0.00751 81941 | 49 13 | 0.86176 99712 | 49 |
| 0.74313 99518 | 1.00424 34799 | 0.00755 14902 | 48 13 | 0.84418 28290 | 48 |
| 0.73134 87191 | 1.00410 98897 | 0.00757 55973 | 47 13 | 0.82659 56867 | 47 |
| 0.71933 47160 | 1.00397 59729 | 0.00759 04823 | 46 13 | 0.80900 85444 | 46 |
| 0.70710 16026 | 1.00384 18928 | 0.00759 61235 | 45 13 | 0.79142 14022 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = 1.5981420021$, $K' = K\sqrt{3} = 2.7680631454$, $E = 1.5441504939$, $E' = 1.076405113$,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01775 71334 | 1 1 | 0.00059 97806 | 1.00000 53258 | 0.01745 10959 |
| 2 | 0.03551 42667 | 2 2 | 0.00119 88113 | 1.00002 12966 | 0.03489 68785 |
| 3 | 0.05327 14001 | 3 3 | 0.00179 63433 | 1.00004 78929 | 0.05233 20359 |
| 4 | 0.07102 85334 | 4 4 | 0.00239 16296 | 1.00008 50825 | 0.06975 12596 |
| 5 | 0.08878 56668 | 5 5 | 0.00298 39265 | 1.00013 28199 | 0.08714 92460 |
| 6 | 0.10654 28002 | 6 6 | 0.00357 24940 | 1.00019 10470 | 0.10452 06976 |
| 7 | 0.12429 99335 | 7 7 | 0.00415 65975 | 1.00025 96929 | 0.12186 03254 |
| 8 | 0.14205 70669 | 8 8 | 0.00473 55081 | 1.00033 86738 | 0.13916 28498 |
| 9 | 0.15981 42002 | 9 9 | 0.00530 85039 | 1.00042 78937 | 0.15642 30024 |
| 10 | 0.17757 13336 | 10 10 | 0.00587 48710 | 1.00052 72438 | 0.17363 55278 |
| 11 | 0.19532 84669 | 11 11 | 0.00643 39044 | 1.00063 66031 | 0.19079 51850 |
| 12 | 0.21308 56003 | 12 12 | 0.00698 49088 | 1.00075 58383 | 0.20789 67491 |
| 13 | 0.23084 27336 | 13 13 | 0.00752 71998 | 1.00088 48041 | 0.22493 50127 |
| 14 | 0.24859 98670 | 14 14 | 0.00806 01044 | 1.00102 33434 | 0.24190 47877 |
| 15 | 0.26635 70004 | 15 15 | 0.00858 29622 | 1.00117 12875 | 0.25880 09068 |
| 16 | 0.28411 41337 | 16 16 | 0.00909 51263 | 1.00132 84561 | 0.27561 82249 |
| 17 | 0.30187 12671 | 17 17 | 0.00959 59638 | 1.00149 46577 | 0.29235 16211 |
| 18 | 0.31962 84004 | 18 18 | 0.01008 48569 | 1.00166 96898 | 0.30899 59997 |
| 19 | 0.33738 55338 | 19 18 | 0.01056 12037 | 1.00185 33392 | 0.32554 62922 |
| 20 | 0.35514 26672 | 20 19 | 0.01102 44188 | 1.00204 53820 | 0.34199 74584 |
| 21 | 0.37289 98005 | 21 20 | 0.01147 39339 | 1.00224 55845 | 0.35834 44886 |
| 22 | 0.39065 69339 | 22 21 | 0.01190 91990 | 1.00245 37025 | 0.37458 24043 |
| 23 | 0.40841 40672 | 23 21 | 0.01232 96827 | 1.00266 94826 | 0.39070 62603 |
| 24 | 0.42617 12006 | 24 22 | 0.01273 48729 | 1.00289 26619 | 0.40671 11462 |
| 25 | 0.44392 83339 | 25 23 | 0.01312 42775 | 1.00312 29684 | 0.42259 21874 |
| 26 | 0.46168 54673 | 26 24 | 0.01349 74251 | 1.00336 01217 | 0.43834 45471 |
| 27 | 0.47944 26006 | 27 25 | 0.01385 38651 | 1.00360 38326 | 0.45396 34276 |
| 28 | 0.49719 97340 | 28 25 | 0.01419 31688 | 1.00385 38044 | 0.46944 40717 |
| 29 | 0.51495 68674 | 29 25 | 0.01451 49297 | 1.00410 97324 | 0.48478 17640 |
| 30 | 0.53271 40007 | 30 26 | 0.01481 87635 | 1.00437 13049 | 0.49997 18327 |
| 31 | 0.55047 11341 | 31 26 | 0.01510 43095 | 1.00463 82031 | 0.51500 96510 |
| 32 | 0.56822 82674 | 32 27 | 0.01537 12298 | 1.00491 01019 | 0.52989 06380 |
| 33 | 0.58598 54008 | 33 27 | 0.01561 92109 | 1.00518 66701 | 0.54461 02607 |
| 34 | 0.60374 25341 | 34 28 | 0.01584 79628 | 1.00546 75706 | 0.55916 40350 |
| 35 | 0.62149 96675 | 35 28 | 0.01605 72204 | 1.00575 24612 | 0.57354 75273 |
| 36 | 0.63925 68009 | 36 28 | 0.01624 67429 | 1.00604 09949 | 0.58775 63556 |
| 37 | 0.65701 39342 | 37 29 | 0.01641 63146 | 1.00633 28201 | 0.60178 61912 |
| 38 | 0.67477 10676 | 38 29 | 0.01656 57446 | 1.00662 75813 | 0.61563 27596 |
| 39 | 0.69252 82009 | 39 29 | 0.01669 48676 | 1.00692 49193 | 0.62929 18421 |
| 40 | 0.71028 53343 | 40 29 | 0.01680 35433 | 1.00722 44718 | 0.64275 92769 |
| 41 | 0.72804 24676 | 41 30 | 0.01689 16569 | 1.00752 58740 | 0.65603 09607 |
| 42 | 0.74579 96010 | 42 30 | 0.01695 91191 | 1.00782 87587 | 0.66910 28494 |
| 43 | 0.76355 67344 | 43 30 | 0.01700 58662 | 1.00813 27567 | 0.68197 09600 |
| 44 | 0.78131 38677 | 44 30 | 0.01703 18597 | 1.00843 74977 | 0.69463 13711 |
| 45 | 0.79907 10011 | 45 30 | 0.01703 70869 | 1.00874 26104 | 0.70708 02248 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.004333420509983$, $\Theta 0 = 0.9913331597$, $HK = 0.5131518035$

| B(r) | C(r) | G(r) | $\psi$ | $F\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.01748 52237 | 0.00000 00000 | 90° 0′ | 1.59814 20021 | 90 |
| 0.99984 76723 | 1.01747 98979 | 0.00058 94801 | 89 1 | 1.58038 48688 | 89 |
| 0.99939 07356 | 1.01746 39271 | 0.00117 82606 | 88 2 | 1.56262 77354 | 88 |
| 0.99862 93293 | 1.01743 73307 | 0.00176 56424 | 87 3 | 1.54487 06021 | 87 |
| 0.99756 36857 | 1.01740 01412 | 0.00235 09281 | 86 4 | 1.52711 34687 | 86 |
| 0.99619 41297 | 1.01735 24037 | 0.00293 34228 | 85 5 | 1.50935 63353 | 85 |
| 0.99452 10792 | 1.01729 41766 | 0.00351 24342 | 84 6 | 1.49159 92020 | 84 |
| 0.99254 50444 | 1.01722 55307 | 0.00408 72741 | 83 7 | 1.47384 20686 | 83 |
| 0.99026 66280 | 1.01714 65496 | 0.00465 72589 | 82 8 | 1.45608 49353 | 82 |
| 0.98768 65251 | 1.01705 73297 | 0.00522 17102 | 81 9 | 1.43832 78019 | 81 |
| 0.98480 55225 | 1.01695 79795 | 0.00577 99557 | 80 10 | 1.42057 06685 | 80 |
| 0.98162 44990 | 1.01684 86202 | 0.00633 13300 | 79 11 | 1.40281 35352 | 79 |
| 0.97814 44248 | 1.01672 93849 | 0.00687 51750 | 78 12 | 1.38505 64019 | 78 |
| 0.97436 63613 | 1.01660 04190 | 0.00741 08412 | 77 13 | 1.36729 92685 | 77 |
| 0.97029 14608 | 1.01646 18796 | 0.00793 76880 | 76 14 | 1.34954 21352 | 76 |
| 0.96592 09661 | 1.01631 39354 | 0.00845 50845 | 75 15 | 1.33178 50018 | 75 |
| 0.96125 62102 | 1.01615 67668 | 0.00896 24102 | 74 16 | 1.31402 78684 | 74 |
| 0.95629 86158 | 1.01599 05651 | 0.00945 90560 | 73 17 | 1.29627 07351 | 73 |
| 0.95104 96947 | 1.01581 55329 | 0.00994 44245 | 72 18 | 1.27851 36017 | 72 |
| 0.94551 10478 | 1.01563 18834 | 0.01041 79308 | 71 18 | 1.26075 64684 | 71 |
| 0.93968 43642 | 1.01543 98405 | 0.01087 90033 | 70 19 | 1.24299 93350 | 70 |
| 0.93357 14207 | 1.01523 96380 | 0.01132 70844 | 69 20 | 1.22524 22016 | 69 |
| 0.92717 40815 | 1.01503 15198 | 0.01176 16310 | 68 20 | 1.20748 50683 | 68 |
| 0.92049 42975 | 1.01481 57396 | 0.01218 21151 | 67 21 | 1.18972 79349 | 67 |
| 0.91353 41057 | 1.01459 25602 | 0.01258 80246 | 66 22 | 1.17197 08016 | 66 |
| 0.90629 56284 | 1.01436 22536 | 0.01297 88640 | 65 23 | 1.15421 36682 | 65 |
| 0.89878 10728 | 1.01412 51003 | 0.01335 41547 | 64 23 | 1.13645 65348 | 64 |
| 0.89099 27303 | 1.01388 13892 | 0.01371 34359 | 63 24 | 1.11869 94015 | 63 |
| 0.88293 29756 | 1.01363 14174 | 0.01405 62649 | 62 25 | 1.10094 22681 | 62 |
| 0.87460 42661 | 1.01337 54893 | 0.01438 22180 | 61 25 | 1.08318 51348 | 61 |
| 0.86600 91414 | 1.01311 39167 | 0.01469 08906 | 60 26 | 1.06542 80014 | 60 |
| 0.85715 02219 | 1.01284 70184 | 0.01498 18982 | 59 26 | 1.04767 08681 | 59 |
| 0.84803 02085 | 1.01257 51195 | 0.01525 48767 | 58 27 | 1.02991 37347 | 58 |
| 0.83865 18817 | 1.01229 85512 | 0.01550 94825 | 57 27 | 1.01215 66014 | 57 |
| 0.82901 81005 | 1.01201 76507 | 0.01574 53939 | 56 28 | 0.99439 94680 | 56 |
| 0.81913 18020 | 1.01173 27599 | 0.01596 23105 | 55 28 | 0.97664 23346 | 55 |
| 0.80899 59997 | 1.01144 42262 | 0.01615 99545 | 54 28 | 0.95888 52013 | 54 |
| 0.79861 37836 | 1.01115 24009 | 0.01633 80704 | 53 29 | 0.94112 80679 | 53 |
| 0.78798 83184 | 1.01085 76397 | 0.01649 64258 | 52 29 | 0.92337 09346 | 52 |
| 0.77712 28430 | 1.01056 03017 | 0.01663 48119 | 51 29 | 0 90561 38012 | 51 |
| 0.76602 06691 | 1.01026 07491 | 0.01675 30432 | 50 29 | 0.88785 66678 | 50 |
| 0.75468 51808 | 1.00995 93468 | 0.01685 09584 | 49 29 | 0.87009 95345 | 49 |
| 0.74311 98330 | 1.00965 64622 | 0.01692 84205 | 48 30 | 0.85234 24011 | 48 |
| 0.73132 81506 | 1.00935 24642 | 0.01698 53170 | 47 30 | 0.83458 52678 | 47 |
| 0.71931 37274 | 1.00904 77232 | 0.01702 15600 | 46 30 | 0.81682 81344 | 46 |
| 0.70708 02248 | 1.00874 26104 | 0.01703 70869 | 45 30 | 0.79907 10011 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | $F\phi$ | r |

K = 1.6200258991, K′ = 2.5045500790, E = 1.5237992053, E′ = 1.118377738

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01800 02878 | 1 2 | 0.00106 89581 | 1.00000 96218 | 0.01744 81883 |
| 2 | 0.03600 05755 | 2 4 | 0.00213 65522 | 1.00003 84757 | 0.03489 10694 |
| 3 | 0.05400 08633 | 3 6 | 0.00320 14202 | 1.00008 65263 | 0.05232 33377 |
| 4 | 0.07200 11511 | 4 7 | 0.00426 22042 | 1.00015 37152 | 0.06973 96909 |
| 5 | 0.09000 14388 | 5 9 | 0.00531 75519 | 1.00023 99605 | 0.08713 48313 |
| 6 | 0.10800 17266 | 6 11 | 0.00636 61189 | 1.00034 51572 | 0.10450 34678 |
| 7 | 0.12600 20144 | 7 13 | 0.00740 65708 | 1.00046 91770 | 0.12184 03169 |
| 8 | 0.14400 23021 | 8 15 | 0.00843 75848 | 1.00061 18689 | 0.13914 01051 |
| 9 | 0.16200 25899 | 9 17 | 0.00945 78515 | 1.00077 30591 | 0.15639 75697 |
| 10 | 0.18000 28777 | 10 19 | 0.01046 60772 | 1.00095 25510 | 0.17360 74610 |
| 11 | 0.19800 31655 | 11 20 | 0.01146 09855 | 1.00115 01262 | 0.19076 45434 |
| 12 | 0.21600 34532 | 12 22 | 0.01244 13188 | 1.00136 55438 | 0.20786 35973 |
| 13 | 0.23400 37410 | 13 24 | 0.01340 58406 | 1.00159 85414 | 0.22489 94205 |
| 14 | 0.25200 40288 | 14 25 | 0.01435 33370 | 1.00184 88351 | 0.24186 68298 |
| 15 | 0.27000 43165 | 15 27 | 0.01528 26180 | 1.00211 61200 | 0.25876 06626 |
| 16 | 0.28800 46043 | 16 28 | 0.01619 25197 | 1.00240 00704 | 0.27557 57786 |
| 17 | 0.30600 48921 | 17 30 | 0.01708 19057 | 1.00270 03405 | 0.29230 70609 |
| 18 | 0.32400 51799 | 18 32 | 0.01794 96683 | 1.00301 65642 | 0.30894 94182 |
| 19 | 0.34200 54676 | 19 33 | 0.01879 47304 | 1.00334 83565 | 0.32549 77855 |
| 20 | 0.36000 57554 | 20 35 | 0.01961 60466 | 1.00369 53131 | 0.34194 71266 |
| 21 | 0.37800 60431 | 21 36 | 0.02041 26046 | 1.00405 70112 | 0.35829 24349 |
| 22 | 0.39600 63309 | 22 37 | 0.02118 34268 | 1.00443 30101 | 0.37452 87349 |
| 23 | 0.41400 66187 | 23 39 | 0.02192 75711 | 1.00482 28518 | 0.39065 10844 |
| 24 | 0.43200 69064 | 24 40 | 0.02264 41321 | 1.00522 60614 | 0.40665 45753 |
| 25 | 0.45000 71942 | 25 41 | 0.02333 22426 | 1.00564 21475 | 0.42253 43354 |
| 26 | 0.46800 74820 | 26 42 | 0.02399 10740 | 1.00607 06033 | 0.43828 55296 |
| 27 | 0.48600 77697 | 27 44 | 0.02461 98378 | 1.00651 09067 | 0.45390 33618 |
| 28 | 0.50400 80575 | 28 45 | 0.02521 77862 | 1.00696 25213 | 0.46938 30761 |
| 29 | 0.52200 83453 | 29 46 | 0.02578 42130 | 1.00742 48968 | 0.48471 99582 |
| 30 | 0.54000 86330 | 30 46 | 0.02631 84541 | 1.00789 74700 | 0.49990 93370 |
| 31 | 0.55800 89208 | 31 47 | 0.02681 98888 | 1.00837 96651 | 0.51494 65858 |
| 32 | 0.57600 92086 | 32 48 | 0.02728 79396 | 1.00887 08946 | 0.52982 71240 |
| 33 | 0.59400 94963 | 33 49 | 0.02772 20732 | 1.00937 05600 | 0.54454 64181 |
| 34 | 0.61200 97841 | 34 50 | 0.02812 18009 | 1.00987 80525 | 0.55909 99835 |
| 35 | 0.63001 00719 | 35 50 | 0.02848 66791 | 1.01039 27539 | 0.57348 33858 |
| 36 | 0.64801 03597 | 36 51 | 0.02881 63091 | 1.01091 40371 | 0.58769 22416 |
| 37 | 0.66601 06474 | 37 51 | 0.02911 03382 | 1.01144 12669 | 0.60172 22208 |
| 38 | 0.68401 09352 | 38 52 | 0.02936 84591 | 1.01197 38011 | 0.61556 90470 |
| 39 | 0.70201 12230 | 39 52 | 0.02959 04103 | 1.01251 09908 | 0.62922 84994 |
| 40 | 0.72001 15107 | 40 53 | 0.02977 59763 | 1.01305 21815 | 0.64269 64140 |
| 41 | 0.73801 17985 | 41 53 | 0.02992 49874 | 1.01359 67138 | 0.65596 86845 |
| 42 | 0.75601 20863 | 42 53 | 0.03003 73198 | 1.01414 39245 | 0.66904 12642 |
| 43 | 0.77401 23740 | 43 53 | 0.03011 28953 | 1.01469 31466 | 0.68191 01665 |
| 44 | 0.79201 26618 | 44 53 | 0.03015 16811 | 1.01524 37112 | 0.69457 14668 |
| 45 | 0.81001 29496 | 45 53 | 0.03015 36896 | 1.01579 49474 | 0.70702 13033 |
| 90 - r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.007774680416442$, $\Theta 0 = 0.9844506465$, $HK = 0.5939185400$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.03158 99246 | 0.00000 00000 | 90° 0′ | 1.62002 58991 | 90 |
| 0.99984 76215 | 1.03158 03027 | 0.00103 62474 | 89 2 | 1.60202 56113 | 89 |
| 0.99939 05327 | 1.03155 14488 | 0.00207 12902 | 88 4 | 1.58402 53236 | 88 |
| 0.99862 88734 | 1.03150 33980 | 0.00310 39250 | 87 6 | 1.56602 50358 | 87 |
| 0.99756 28767 | 1.03143 62088 | 0.00413 29509 | 86 7 | 1.54802 47480 | 86 |
| 0.99619 28686 | 1.03134 99632 | 0.00515 71704 | 85 9 | 1.53002 44603 | 85 |
| 0.99451 92682 | 1.03124 47661 | 0.00617 53910 | 84 11 | 1.51202 41725 | 84 |
| 0.99254 25876 | 1.03112 07458 | 0.00718 64259 | 83 13 | 1.49402 38847 | 83 |
| 0.99026 34315 | 1.03097 80534 | 0.00818 90957 | 82 15 | 1.47602 35970 | 82 |
| 0.98768 24970 | 1.03081 68627 | 0.00918 22293 | 81 16 | 1.45802 33092 | 81 |
| 0.98480 05736 | 1.03063 73701 | 0.01016 46651 | 80 18 | 1.44002 30214 | 80 |
| 0.98161 85429 | 1.03043 97942 | 0.01113 52523 | 79 20 | 1.42202 27337 | 79 |
| 0.97813 73781 | 1.03022 43759 | 0.01209 28519 | 78 22 | 1.40402 24459 | 78 |
| 0.97435 81442 | 1.02999 13775 | 0.01303 63381 | 77 23 | 1.38602 21581 | 77 |
| 0.97028 19968 | 1.02974 10829 | 0.01396 45994 | 76 25 | 1.36802 18704 | 76 |
| 0.96591 01827 | 1.02947 37972 | 0.01487 65396 | 75 27 | 1.35002 15826 | 75 |
| 0.96124 40390 | 1.02918 98458 | 0.01577 10793 | 74 28 | 1.33202 12948 | 74 |
| 0.95628 49924 | 1.02888 95748 | 0.01664 71568 | 73 30 | 1.31402 10070 | 73 |
| 0.95103 45595 | 1.02857 33501 | 0.01750 37292 | 72 31 | 1.29602 07193 | 72 |
| 0.94549 43456 | 1.02824 15568 | 0.01833 97739 | 71 33 | 1.27802 04315 | 71 |
| 0.93966 60449 | 1.02789 45992 | 0.01915 42895 | 70 34 | 1.26002 01437 | 70 |
| 0.93355 14391 | 1.02753 28994 | 0.01994 62967 | 69 36 | 1.24201 98560 | 69 |
| 0.92715 23977 | 1.02715 69001 | 0.02071 48399 | 68 37 | 1.22401 95682 | 68 |
| 0.92047 08768 | 1.02676 70574 | 0.02145 89881 | 67 38 | 1.20601 92804 | 67 |
| 0.91350 89187 | 1.02636 38468 | 0.02217 78360 | 66 40 | 1.18801 89927 | 66 |
| 0.90626 86515 | 1.02594 77596 | 0.02287 05049 | 65 41 | 1.17001 87049 | 65 |
| 0.89875 22880 | 1.02551 93029 | 0.02353 61442 | 64 42 | 1.15201 84171 | 64 |
| 0.89096 21252 | 1.02507 89985 | 0.02417 39320 | 63 43 | 1.13401 81294 | 63 |
| 0.88290 05436 | 1.02462 73829 | 0.02478 30767 | 62 44 | 1.11601 78416 | 62 |
| 0.87457 00067 | 1.02416 50064 | 0.02536 28172 | 61 45 | 1.09801 75538 | 61 |
| 0.86597 30595 | 1.02369 24323 | 0.02591 24248 | 60 46 | 1.08001 72661 | 60 |
| 0.85711 23285 | 1.02321 02363 | 0.02643 12037 | 59 47 | 1.06201 69783 | 59 |
| 0.84799 05205 | 1.02271 90060 | 0.02691 84920 | 58 48 | 1.04401 66905 | 58 |
| 0.83861 04218 | 1.02221 93398 | 0.02737 36626 | 57 49 | 1.02601 64028 | 57 |
| 0.82897 48973 | 1.02171 18465 | 0.02779 61243 | 56 49 | 1.00801 61150 | 56 |
| 0.81908 68896 | 1.02119 71444 | 0.02818 53227 | 55 50 | 0.99001 58272 | 55 |
| 0.80894 94182 | 1.02067 58606 | 0.02854 07409 | 54 51 | 0.97201 55395 | 54 |
| 0.79856 55784 | 1.02014 86302 | 0.02886 19001 | 53 51 | 0.95401 52517 | 53 |
| 0.78793 85407 | 1.01961 60955 | 0.02914 83611 | 52 52 | 0.93601 49639 | 52 |
| 0.77707 15491 | 1.01907 89054 | 0.02939 97245 | 51 52 | 0.91801 46761 | 51 |
| 0.76596 79209 | 1.01853 77143 | 0.02961 56313 | 50 53 | 0.90001 43884 | 50 |
| 0.75463 10450 | 1.01799 31816 | 0.02979 57642 | 49 53 | 0.88201 41006 | 49 |
| 0.74306 43814 | 1.01744 59707 | 0.02993 98477 | 48 53 | 0.86401 38129 | 48 |
| 0.73127 14598 | 1.01689 67484 | 0.03004 76489 | 47 53 | 0.84601 35251 | 47 |
| 0.71925 58784 | 1.01634 61837 | 0.03011 89783 | 46 53 | 0.82801 32373 | 46 |
| 0.70702 13033 | 1.01579 49474 | 0.03015 36896 | 45 53 | 0.81001 29496 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 1.6489952185, K′ = 2.3087867982, E = 1.4981149284, E′ = 1.1638279645,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01832 21691 | 1 3 | 0.00167 60815 | 1.00001 53565 | 0.01744 18591 |
| 2 | 0.03664 43382 | 2 6 | 0.00334 99667 | 1.00006 14074 | 0.03487 84245 |
| 3 | 0.05496 65073 | 3 9 | 0.00501 94629 | 1.00013 80964 | 0.05230 44041 |
| 4 | 0.07328 86764 | 4 12 | 0.00668 23842 | 1.00024 53303 | 0.06971 45088 |
| 5 | 0.09161 08455 | 5 15 | 0.00833 65551 | 1.00038 29783 | 0.08710 34544 |
| 6 | 0.10993 30145 | 6 18 | 0.00997 98139 | 1.00055 08728 | 0.10446 59627 |
| 7 | 0.12825 51836 | 7 21 | 0.01161 00163 | 1.00074 88092 | 0.12179 67635 |
| 8 | 0.14657 73527 | 8 24 | 0.01322 50382 | 1.00097 65463 | 0.13909 05958 |
| 9 | 0.16489 95218 | 9 26 | 0.01482 27797 | 1.00123 38067 | 0.15634 22095 |
| 10 | 0.18322 16909 | 10 29 | 0.01640 11677 | 1.00152 02770 | 0.17354 63669 |
| 11 | 0.20154 38600 | 11 32 | 0.01795 81596 | 1.00183 56081 | 0.19069 78446 |
| 12 | 0.21986 60291 | 12 35 | 0.01949 17458 | 1.00217 94159 | 0.20779 14345 |
| 13 | 0.23818 81982 | 13 37 | 0.02099 99533 | 1.00255 12815 | 0.22482 19454 |
| 14 | 0.25651 03673 | 14 40 | 0.02248 08485 | 1.00295 07519 | 0.24178 42052 |
| 15 | 0.27483 25364 | 15 43 | 0.02393 25396 | 1.00337 73404 | 0.25867 30615 |
| 16 | 0.29315 47055 | 16 45 | 0.02535 31798 | 1.00383 05272 | 0.27548 33838 |
| 17 | 0.31147 68746 | 17 48 | 0.02674 09700 | 1.00430 97603 | 0.29221 00649 |
| 18 | 0.32979 90437 | 18 50 | 0.02809 41609 | 1.00481 44557 | 0.30884 80221 |
| 19 | 0.34812 12128 | 19 53 | 0.02941 10555 | 1.00534 39986 | 0.32539 21991 |
| 20 | 0.36644 33819 | 20 56 | 0.03069 00118 | 1.00589 77438 | 0.34183 75673 |
| 21 | 0.38476 55510 | 21 57 | 0.03192 94445 | 1.00647 50167 | 0.35817 91274 |
| 22 | 0.40308 77201 | 22 59 | 0.03312 78272 | 1.00707 51140 | 0.37441 19107 |
| 23 | 0.42140 98892 | 24 1 | 0.03428 36945 | 1.00769 73046 | 0.39053 09808 |
| 24 | 0.43973 28582 | 25 3 | 0.03539 56434 | 1.00834 08304 | 0.40653 14352 |
| 25 | 0.45805 42273 | 26 5 | 0.03646 23352 | 1.00900 49074 | 0.42240 84064 |
| 26 | 0.47637 63964 | 27 7 | 0.03748 24970 | 1.00968 87266 | 0.43815 70635 |
| 27 | 0.49469 85655 | 28 9 | 0.03845 49232 | 1.01039 14548 | 0.45377 26140 |
| 28 | 0.51302 07346 | 29 11 | 0.03937 84764 | 1.01111 22358 | 0.46925 03045 |
| 29 | 0.53134 29037 | 30 12 | 0.04025 20886 | 1.01185 01916 | 0.48458 54231 |
| 30 | 0.54966 50728 | 31 14 | 0.04107 47627 | 1.01260 44231 | 0.49977 32999 |
| 31 | 0.56798 72419 | 32 15 | 0.04184 55726 | 1.01337 40113 | 0.51480 93092 |
| 32 | 0.58630 94110 | 33 16 | 0.04256 36643 | 1.01415 80186 | 0.52968 88703 |
| 33 | 0.60463 15801 | 34 18 | 0.04322 82564 | 1.01495 54899 | 0.54440 74492 |
| 34 | 0.62295 37492 | 35 19 | 0.04383 86406 | 1.01576 54535 | 0.55896 05600 |
| 35 | 0.64127 59183 | 36 20 | 0.04439 41821 | 1.01658 69227 | 0.57334 37662 |
| 36 | 0.65959 80874 | 37 21 | 0.04489 43196 | 1.01741 88967 | 0.58755 26819 |
| 37 | 0.67792 02565 | 38 22 | 0.04533 85655 | 1.01826 03617 | 0.60158 29737 |
| 38 | 0.69624 24256 | 39 23 | 0.04572 65058 | 1.01911 02927 | 0.61543 03611 |
| 39 | 0.71456 45947 | 40 23 | 0.04605 78000 | 1.01996 76540 | 0.62909 06189 |
| 40 | 0.73288 67638 | 41 23 | 0.04633 21809 | 1.02083 14013 | 0.64255 95777 |
| 41 | 0.75120 89328 | 42 24 | 0.04654 94543 | 1.02170 04820 | 0.65583 31255 |
| 42 | 0.76953 11019 | 43 24 | 0.04670 94981 | 1.02257 38374 | 0.66890 72089 |
| 43 | 0.78785 32710 | 44 24 | 0.04681 22622 | 1.02345 04035 | 0.68177 78347 |
| 44 | 0.80617 54401 | 45 24 | 0.04685 77678 | 1.02432 91122 | 0.69444 10704 |
| 45 | 0.82449 76092 | 46 24 | 0.04684 61065 | 1.02520 88930 | 0.70689 30463 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 25°$

$q = 0.012294560527181$, $\Theta 0 = 0.975410924642$, $HK = 0.666076159327$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 - r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.05041 79735 | 0.00000 00000 | 90° 0′ | 1.64899 52185 | 90 |
| 0.99984 75111 | 1.05040 26167 | 0.00159 57045 | 89 3 | 1.63067 30494 | 89 |
| 0.99939 00912 | 1.05035 65652 | 0.00318 96046 | 88 6 | 1.61235 08803 | 88 |
| 0.99862 78812 | 1.05027 98750 | 0.00477 98977 | 87 9 | 1.59402 87112 | 87 |
| 0.99756 11158 | 1.05017 26395 | 0.00636 47840 | 86 12 | 1.57570 65421 | 86 |
| 0.99619 01235 | 1.05003 49895 | 0.00794 24686 | 85 15 | 1.55738 43730 | 85 |
| 0.99451 53263 | 1.04986 70926 | 0.00951 11627 | 84 17 | 1.53906 22039 | 84 |
| 0.99253 72400 | 1.04966 91533 | 0.01106 90855 | 83 20 | 1.52074 00348 | 83 |
| 0.99025 64734 | 1.04944 14129 | 0.01261 44653 | 82 23 | 1.50241 78657 | 82 |
| 0.98767 37287 | 1.04918 41489 | 0.01414 55416 | 81 26 | 1.48409 56966 | 81 |
| 0.98478 98010 | 1.04889 76746 | 0.01566 05663 | 80 29 | 1.46577 35275 | 80 |
| 0.98160 55779 | 1.04858 23391 | 0.01715 78054 | 79 31 | 1.44745 13584 | 79 |
| 0.97812 20395 | 1.04823 85265 | 0.01863 55407 | 78 34 | 1.42912 91893 | 78 |
| 0.97434 02576 | 1.04786 66559 | 0.02009 20712 | 77 37 | 1.41080 70202 | 77 |
| 0.97026 13962 | 1.04746 71802 | 0.02152 57149 | 76 39 | 1.39248 48511 | 76 |
| 0.96588 67101 | 1.04704 05862 | 0.02293 48102 | 75 42 | 1.37416 26821 | 75 |
| 0.96121 75452 | 1.04658 73936 | 0.02431 77177 | 74 44 | 1.35584 05130 | 74 |
| 0.95625 53377 | 1.04610 81546 | 0.02567 28218 | 73 47 | 1.33751 83439 | 73 |
| 0.95100 16139 | 1.04560 34530 | 0.02699 85322 | 72 49 | 1.31919 61748 | 72 |
| 0.94545 79893 | 1.04507 39038 | 0.02829 32857 | 71 52 | 1.30087 40057 | 71 |
| 0.93962 61686 | 1.04452 01522 | 0.02955 55477 | 70 54 | 1.28255 18366 | 70 |
| 0.93350 79444 | 1.04394 28728 | 0.03078 38140 | 69 56 | 1.26422 96675 | 69 |
| 0.92710 51976 | 1.04334 27690 | 0.03197 66123 | 68 58 | 1.24590 74984 | 68 |
| 0.92041 98958 | 1.04272 05719 | 0.03313 25038 | 68 0 | 1.22758 53293 | 67 |
| 0.91345 40932 | 1.04207 70396 | 0.03425 00853 | 67 2 | 1.20926 31602 | 66 |
| 0.90620 99299 | 1.04141 29561 | 0.03532 79902 | 66 4 | 1.19094 09911 | 65 |
| 0.89868 96309 | 1.04072 91305 | 0.03636 48907 | 65 6 | 1.17261 88220 | 64 |
| 0.89089 55058 | 1.04002 63960 | 0.03735 94992 | 64 8 | 1.15429 66529 | 63 |
| 0.88282 99477 | 1.03930 56088 | 0.03831 05700 | 63 10 | 1.13597 44838 | 62 |
| 0.87449 54326 | 1.03856 76470 | 0.03921 69009 | 62 11 | 1.11765 23147 | 61 |
| 0.86589 45184 | 1.03781 34098 | 0.04007 73349 | 61 13 | 1.09933 01456 | 60 |
| 0.85702 98444 | 1.03704 38161 | 0.04089 07619 | 60 14 | 1.08100 79765 | 59 |
| 0.84790 41300 | 1.03625 98035 | 0.04165 61200 | 59 16 | 1.06268 58075 | 58 |
| 0.83852 01744 | 1.03546 23272 | 0.04237 23976 | 58 17 | 1.04436 36384 | 57 |
| 0.82888 08549 | 1.03465 23588 | 0.04303 86345 | 57 18 | 1.02604 14693 | 56 |
| 0.81898 91269 | 1.03383 08852 | 0.04365 39236 | 56 19 | 1.00771 93002 | 55 |
| 0.80884 80221 | 1.03299 89073 | 0.04421 74127 | 55 20 | 0.98939 71311 | 54 |
| 0.79846 06482 | 1.03215 74386 | 0.04472 83056 | 54 21 | 0.97107 49620 | 53 |
| 0.78783 01874 | 1.03130 75044 | 0.04518 58637 | 53 22 | 0.95275 27929 | 52 |
| 0.77695 98956 | 1.03045 01401 | 0.04558 94076 | 52 22 | 0.93443 06238 | 51 |
| 0.76585 31015 | 1.02958 63905 | 0.04593 83183 | 51 23 | 0.91610 84547 | 50 |
| 0.75451 32053 | 1.02871 73077 | 0.04623 20386 | 50 24 | 0.89778 62856 | 49 |
| 0.74294 36775 | 1.02784 39507 | 0.04647 00744 | 49 24 | 0.87946 41165 | 48 |
| 0.73114 80583 | 1.02696 73835 | 0.04665 19961 | 48 24 | 0.86114 19474 | 47 |
| 0.71912 99561 | 1.02608 86741 | 0.04677 74393 | 47 24 | 0.84281 97783 | 46 |
| 0.70689 30463 | 1.02520 88930 | 0.04684 61065 | 46 24 | 0.82449 76092 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 1.6857503548, K′ = 2.1565156475, E = 1.4674622093 E′ = 1.211056028,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01873 05595 | 1 4 | 0.00242 48763 | 1.00002 27125 | 0.01742 98716 |
| 2 | 0.03746 11190 | 2 9 | 0.00484 64683 | 1.00009 08222 | 0.03485 44751 |
| 3 | 0.05619 16785 | 3 13 | 0.00726 14977 | 1.00020 42462 | 0.05226 85438 |
| 4 | 0.07492 22380 | 4 18 | 0.00966 66975 | 1.00036 28463 | 0.06966 68140 |
| 5 | 0.09365 27975 | 5 22 | 0.01205 88178 | 1.00056 64294 | 0.08704 40267 |
| 6 | 0.11238 33570 | 6 26 | 0.01443 46319 | 1.00081 47472 | 0.10439 49285 |
| 7 | 0.13111 39165 | 7 30 | 0.01679 09412 | 1.00110 74975 | 0.12171 42736 |
| 8 | 0.14984 44760 | 8 35 | 0.01912 45813 | 1.00144 43235 | 0.13899 68254 |
| 9 | 0.16857 50355 | 9 39 | 0.02143 24269 | 1.00182 48148 | 0.15623 73574 |
| 10 | 0.18730 55950 | 10 43 | 0.02371 13976 | 1.00224 85079 | 0.17343 06551 |
| 11 | 0.20603 61545 | 11 47 | 0.02595 84626 | 1.00271 48868 | 0.19057 15175 |
| 12 | 0.22476 67140 | 12 51 | 0.02817 06459 | 1.00322 33830 | 0.20765 47584 |
| 13 | 0.24349 72734 | 13 55 | 0.03034 50312 | 1.00377 33773 | 0.22467 52081 |
| 14 | 0.26222 78329 | 14 59 | 0.03247 87664 | 1.00436 41996 | 0.24162 77146 |
| 15 | 0.28095 83924 | 16 3 | 0.03456 90685 | 1.00499 51300 | 0.25850 71454 |
| 16 | 0.29968 89519 | 17 6 | 0.03661 32272 | 1.00566 54000 | 0.27530 83886 |
| 17 | 0.31841 95114 | 18 10 | 0.03860 86097 | 1.00637 41929 | 0.29202 63549 |
| 18 | 0.33715 00709 | 19 14 | 0.04055 26642 | 1.00712 06453 | 0.30865 59785 |
| 19 | 0.35588 06304 | 20 17 | 0.04244 29236 | 1.00790 38477 | 0.32519 22190 |
| 20 | 0.37461 11899 | 21 20 | 0.04427 70092 | 1.00872 28461 | 0.34163 00625 |
| 21 | 0.39334 17494 | 22 23 | 0.04605 26335 | 1.00957 66426 | 0.35796 45236 |
| 22 | 0.41207 23089 | 23 27 | 0.04776 76034 | 1.01046 41971 | 0.37419 06461 |
| 23 | 0.43080 28684 | 24 30 | 0.04941 98229 | 1.01138 44282 | 0.39030 35051 |
| 24 | 0.44953 34279 | 25 33 | 0.05100 72958 | 1.01233 62150 | 0.40629 82084 |
| 25 | 0.46826 39874 | 26 36 | 0.05252 81275 | 1.01331 83978 | 0.42216 98975 |
| 26 | 0.48699 45469 | 27 38 | 0.05398 05273 | 1.01432 97800 | 0.43791 37495 |
| 27 | 0.50572 51064 | 28 41 | 0.05536 28100 | 1.01536 91295 | 0.45352 49782 |
| 28 | 0.52445 56659 | 29 43 | 0.05667 33976 | 1.01643 51800 | 0.46899 88358 |
| 29 | 0.54318 62254 | 30 46 | 0.05791 08204 | 1.01752 66329 | 0.48433 06142 |
| 30 | 0.56191 67849 | 31 48 | 0.05907 37181 | 1.01864 21583 | 0.49951 56464 |
| 31 | 0.58064 73444 | 32 50 | 0.06016 08407 | 1.01978 03972 | 0.51454 93080 |
| 32 | 0.59937 79039 | 33 52 | 0.06117 10486 | 1.02093 99629 | 0.52942 70185 |
| 33 | 0.61810 84634 | 34 54 | 0.06210 33138 | 1.02211 94428 | 0.54414 42428 |
| 34 | 0.63683 90229 | 35 55 | 0.06295 67191 | 1.02331 73997 | 0.55869 64925 |
| 35 | 0.65556 95824 | 36 56 | 0.06373 04587 | 1.02453 23743 | 0.57307 93274 |
| 36 | 0.67430 01419 | 37 58 | 0.06442 38375 | 1.02576 28863 | 0.58728 83566 |
| 37 | 0.69303 07014 | 38 59 | 0.06503 62710 | 1.02700 74365 | 0.60131 92403 |
| 38 | 0.71176 12609 | 40 0 | 0.06556 72843 | 1.02826 45087 | 0.61516 76907 |
| 39 | 0.73049 18204 | 41 1 | 0.06601 65112 | 1.02953 25714 | 0.62882 94738 |
| 40 | 0.74922 23799 | 42 2 | 0.06638 36938 | 1.03081 00797 | 0.64230 04103 |
| 41 | 0.76795 29394 | 43 3 | 0.06666 86806 | 1.03209 54771 | 0.65557 63772 |
| 42 | 0.78668 34989 | 44 3 | 0.06687 14255 | 1.03338 71976 | 0.66865 33089 |
| 43 | 0.80541 40584 | 45 3 | 0.06699 19865 | 1.03468 36674 | 0.68152 71988 |
| 44 | 0.82414 46179 | 46 4 | 0.06703 05237 | 1.03598 33070 | 0.69419 41003 |
| 45 | 0.84287 51774 | 47 3 | 0.06698 72981 | 1.03728 45330 | 0.70665 01282 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 30°$

$q = 0.017972387008967$, $\Theta 0 = 0.9640554346$, $HK = 0.7325237222$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.07456 99318 | 0.00000 00000 | 90° 0′ | 1.68575 03548 | 90 |
| 0.99984 73018 | 1.07454 72183 | 0.00225 68053 | 89 4 | 1.66701 97953 | 89 |
| 0.99938 92548 | 1.07447 91054 | 0.00451 11469 | 88 8 | 1.64828 92358 | 88 |
| 0.99862 60018 | 1.07436 56761 | 0.00676 05625 | 87 13 | 1.62955 86763 | 87 |
| 0.99755 77806 | 1.07420 70687 | 0.00900 25936 | 86 17 | 1.61082 81168 | 86 |
| 0.99618 49242 | 1.07400 34764 | 0.01123 47869 | 85 21 | 1.59209 75573 | 85 |
| 0.99450 78603 | 1.07375 51471 | 0.01345 46957 | 84 25 | 1.57336 69978 | 84 |
| 0.99252 71115 | 1.07346 23837 | 0.01565 98823 | 83 29 | 1.55463 64383 | 83 |
| 0.99024 32948 | 1.07312 55426 | 0.01784 79196 | 82 33 | 1.53590 58788 | 82 |
| 0.98765 71218 | 1.07274 50344 | 0.02001 63924 | 81 38 | 1.51717 53193 | 81 |
| 0.98476 93979 | 1.07232 13226 | 0.02216 28998 | 80 42 | 1.49844 47598 | 80 |
| 0.98158 10224 | 1.07185 49236 | 0.02428 50568 | 79 46 | 1.47971 42003 | 79 |
| 0.97809 29880 | 1.07134 64055 | 0.02638 04961 | 78 49 | 1.46098 36408 | 78 |
| 0.97430 63806 | 1.07079 63881 | 0.02844 68702 | 77 53 | 1.44225 30813 | 77 |
| 0.97022 23787 | 1.07020 55414 | 0.03048 18529 | 76 57 | 1.42352 25218 | 76 |
| 0.96584 22530 | 1.06957 45853 | 0.03248 31417 | 76 1 | 1.40479 19623 | 75 |
| 0.96116 73661 | 1.06890 42887 | 0.03444 84594 | 75 4 | 1.38606 14028 | 74 |
| 0.95619 91719 | 1.06819 54682 | 0.03637 55563 | 74 8 | 1.36733 08433 | 73 |
| 0.95093 92151 | 1.06744 89874 | 0.03826 22123 | 73 12 | 1.34860 02839 | 72 |
| 0.94538 91306 | 1.06666 57559 | 0.04010 62389 | 72 15 | 1.32986 97244 | 71 |
| 0.93955 06429 | 1.06584 67280 | 0.04190 54809 | 71 18 | 1.31113 91649 | 70 |
| 0.93342 55657 | 1.06499 29016 | 0.04365 78194 | 70 22 | 1.29240 86054 | 69 |
| 0.92701 58009 | 1.06410 53170 | 0.04536 11731 | 69 25 | 1.27367 80459 | 68 |
| 0.92032 33381 | 1.06318 50556 | 0.04701 35012 | 68 28 | 1.25494 74864 | 67 |
| 0.91335 02539 | 1.06223 32387 | 0.04861 28052 | 67 31 | 1.23621 69269 | 66 |
| 0.90609 87113 | 1.06125 10260 | 0.05015 71313 | 66 34 | 1.21748 63674 | 65 |
| 0.89857 09587 | 1.06023 96142 | 0.05164 45728 | 65 36 | 1.19875 58079 | 64 |
| 0.89076 93291 | 1.05920 02357 | 0.05307 32725 | 64 39 | 1.18002 52484 | 63 |
| 0.88269 62394 | 1.05813 41567 | 0.05444 14248 | 63 41 | 1.16129 46889 | 62 |
| 0.87435 41897 | 1.05704 26763 | 0.05574 72783 | 62 44 | 1.14256 41294 | 61 |
| 0.86574 57620 | 1.05592 71242 | 0.05698 91384 | 61 46 | 1.12383 35699 | 60 |
| 0.85687 36199 | 1.05478 88596 | 0.05816 53694 | 60 48 | 1.10510 30104 | 59 |
| 0.84774 05068 | 1.05362 92695 | 0.05927 43970 | 59 50 | 1.08637 24509 | 58 |
| 0.83834 92461 | 1.05244 97665 | 0.06031 47110 | 58 52 | 1.06764 18914 | 57 |
| 0.82870 27391 | 1.05125 17878 | 0.06128 48679 | 57 54 | 1.04891 13319 | 56 |
| 0.81880 39648 | 1.05003 67930 | 0.06218 34927 | 56 55 | 1.03018 07724 | 55 |
| 0.80865 59785 | 1.04880 62525 | 0.06300 92824 | 55 57 | 1.01145 02129 | 54 |
| 0.79826 19108 | 1.04756 16953 | 0.06376 10074 | 54 58 | 0.99271 96534 | 53 |
| 0.78762 49668 | 1.04630 46080 | 0.06443 75150 | 53 59 | 0.97398 90939 | 52 |
| 0.77674 84245 | 1.04503 65320 | 0.06503 77310 | 53 0 | 0.95525 85344 | 51 |
| 0.76563 56343 | 1.04375 90125 | 0.06556 06627 | 52 1 | 0.93652 79749 | 50 |
| 0.75429 00174 | 1.04247 36057 | 0.06600 54011 | 51 2 | 0.91779 74154 | 49 |
| 0.74271 50649 | 1.04118 18779 | 0.06637 11230 | 50 3 | 0.89906 68559 | 48 |
| 0.73091 43366 | 1.03988 54029 | 0.06665 70938 | 49 3 | 0.88033 62964 | 47 |
| 0.71889 14599 | 1.03858 57601 | 0.06686 26693 | 48 3 | 0.86160 57369 | 46 |
| 0.70665 01282 | 1.03728 45330 | 0.06698 72981 | 47 3 | 0.84287 51774 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 1.7312451757, K′ = 2.0347153122, E = 1.4322909693, E′ = 1.2586796248,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01923 60575 | 1 6 | 0.00332 09329 | 1.00003 19451 | 0.01740 91115 |
| 2 | 0.03847 21150 | 2 12 | 0.00663 71847 | 1.00012 77415 | 0.03481 29991 |
| 3 | 0.05770 81725 | 3 18 | 0.00994 40836 | 1.00028 72724 | 0.05220 64403 |
| 4 | 0.07694 42300 | 4 24 | 0.01323 69759 | 1.00051 03436 | 0.06958 42154 |
| 5 | 0.09618 02875 | 5 30 | 0.01651 12357 | 1.00079 66833 | 0.08694 11086 |
| 6 | 0.11541 63450 | 6 36 | 0.01976 22733 | 1.00114 59427 | 0.10427 19100 |
| 7 | 0.13465 24025 | 7 42 | 0.02298 55446 | 1.00155 76965 | 0.12157 14162 |
| 8 | 0.15388 84600 | 8 48 | 0.02617 65594 | 1.00203 14429 | 0.13883 44322 |
| 9 | 0.17312 45176 | 9 54 | 0.02933 08900 | 1.00256 66050 | 0.15605 57726 |
| 10 | 0.19236 05751 | 11 0 | 0.03244 41797 | 1.00316 25308 | 0.17323 02632 |
| 11 | 0.21159 66326 | 12 5 | 0.03551 21508 | 1.00381 84944 | 0.19035 27418 |
| 12 | 0.23083 26901 | 13 11 | 0.03853 06122 | 1.00453 36968 | 0.20741 80603 |
| 13 | 0.25006 87476 | 14 16 | 0.04149 54668 | 1.00530 72668 | 0.22442 10857 |
| 14 | 0.26930 48051 | 15 22 | 0.04440 27192 | 1.00613 82620 | 0.24135 67013 |
| 15 | 0.28854 08626 | 16 27 | 0.04724 84818 | 1.00702 56701 | 0.25821 98088 |
| 16 | 0.30777 69201 | 17 32 | 0.05002 89819 | 1.00796 84103 | 0.27500 53288 |
| 17 | 0.32701 29776 | 18 37 | 0.05274 05671 | 1.00896 53340 | 0.29170 82026 |
| 18 | 0.34624 90351 | 19 42 | 0.05537 97118 | 1.01001 52268 | 0.30832 33939 |
| 19 | 0.36548 50926 | 20 47 | 0.05794 30217 | 1.01111 68099 | 0.32484 58897 |
| 20 | 0.38472 11501 | 21 52 | 0.06042 72392 | 1.01226 87413 | 0 34127 07019 |
| 21 | 0.40395 72077 | 22 56 | 0.06282 92476 | 1.01346 96177 | 0.35759 28687 |
| 22 | 0.42319 32652 | 24 0 | 0.06514 60751 | 1.01471 79763 | 0.37380 74559 |
| 23 | 0.44242 93227 | 25 5 | 0.06737 48988 | 1.01601 22964 | 0.38990 95585 |
| 24 | 0.46166 53802 | 26 9 | 0.06951 30473 | 1.01735 10012 | 0.40589 43019 |
| 25 | 0.48090 14377 | 27 13 | 0.07155 80036 | 1.01873 24599 | 0.42175 68435 |
| 26 | 0.50013 74952 | 28 16 | 0.07350 74079 | 1.02015 49897 | 0.43749 23737 |
| 27 | 0.51937 35527 | 29 20 | 0.07535 90588 | 1.02161 68576 | 0.45309 61179 |
| 28 | 0.53860 96102 | 30 23 | 0.07711 09151 | 1.02311 62828 | 0.46856 33375 |
| 29 | 0.55784 56677 | 31 27 | 0.07876 10969 | 1.02465 14386 | 0.48388 93314 |
| 30 | 0.57708 17252 | 32 30 | 0.08030 78862 | 1.02622 04548 | 0.49906 94371 |
| 31 | 0.59631 77827 | 33 32 | 0.08174 97274 | 1.02782 14201 | 0.51409 90330 |
| 32 | 0.61555 38402 | 34 35 | 0.08308 52267 | 1.02945 23841 | 0.52897 35386 |
| 33 | 0.63478 98977 | 35 37 | 0.08431 31523 | 1.03111 13599 | 0.54368 84170 |
| 34 | 0.65402 59552 | 36 40 | 0.08543 24331 | 1.03279 63263 | 0.55823 91754 |
| 35 | 0.67326 20128 | 37 42 | 0.08644 21580 | 1.03450 52308 | 0.57262 13672 |
| 36 | 0.69249 80703 | 38 43 | 0.08734 15741 | 1.03623 59914 | 0.58683 05928 |
| 37 | 0.71173 41278 | 39 45 | 0.08813 00853 | 1.03798 64996 | 0.60086 25017 |
| 38 | 0.73097 01853 | 40 46 | 0.08880 72502 | 1.03975 46228 | 0.61471 27930 |
| 39 | 0.75020 62428 | 41 48 | 0.08937 27798 | 1.04153 82068 | 0.62837 72177 |
| 40 | 0.76944 23003 | 42 49 | 0.08982 65352 | 1.04333 50787 | 0.64185 15792 |
| 41 | 0.78867 83578 | 43 49 | 0.09016 85246 | 1.04514 30495 | 0.65513 17355 |
| 42 | 0.80791 44153 | 44 50 | 0.09039 89009 | 1.04695 99164 | 0.66821 35999 |
| 43 | 0.82715 04728 | 45 50 | 0.09051 79579 | 1.04878 34660 | 0.68109 31428 |
| 44 | 0.84638 65303 | 46 51 | 0.09052 61280 | 1.05061 14765 | 0.69376 63926 |
| 45 | 0.86562 25878 | 47 51 | 0.09042 39779 | 1.05244 17208 | 0.70622 94378 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.024915062523981$, $\Theta 0 = 0.9501706456$, HK $= 0.7950876364$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90−r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.10488 66859 | 0.00000 00000 | 90° 0′ | 1.73124 51757 | 90 |
| 0.99984 69394 | 1.10485 47369 | 0.00300 62320 | 89 6 | 1.71200 91181 | 89 |
| 0.99938 78065 | 1.10475 89287 | 0.00600 93218 | 88 12 | 1.69277 30606 | 88 |
| 0.99862 27471 | 1.10459 93781 | 0.00900 61288 | 87 17 | 1.67353 70031 | 87 |
| 0.99755 20048 | 1.10437 62795 | 0.01199 35156 | 86 23 | 1.65430 09456 | 86 |
| 0.99617 59200 | 1.10408 99048 | 0.01496 83495 | 85 29 | 1.63506 48881 | 85 |
| 0.99449 49305 | 1.10374 06029 | 0.01792 75043 | 84 35 | 1.61582 88306 | 84 |
| 0.99250 95707 | 1.10332 87996 | 0.02086 78620 | 83 40 | 1.59659 27731 | 83 |
| 0.99022 04719 | 1.10285 49965 | 0.02378 63141 | 82 46 | 1.57735 67156 | 82 |
| 0.98762 83615 | 1.10231 97711 | 0.02667 97640 | 81 51 | 1.55812 06581 | 81 |
| 0.98473 40633 | 1.10172 37756 | 0.02954 51279 | 80 57 | 1.53888 46006 | 80 |
| 0.98153 84966 | 1.10106 77362 | 0.03237 93372 | 80 2 | 1.51964 85431 | 79 |
| 0.97804 26763 | 1.10035 24524 | 0.03517 93404 | 79 8 | 1.50041 24856 | 78 |
| 0.97424 77117 | 1.09957 87957 | 0.03794 21046 | 78 13 | 1.48117 64281 | 77 |
| 0.97015 48073 | 1.09874 77089 | 0.04066 46178 | 77 19 | 1.46194 03706 | 76 |
| 0.96576 52612 | 1.09786 02047 | 0.04334 38907 | 76 24 | 1.44270 43130 | 75 |
| 0.96108 04649 | 1.09691 73646 | 0.04597 69592 | 75 29 | 1.42346 82555 | 74 |
| 0.95610 19028 | 1.09592 03375 | 0.04856 08861 | 74 34 | 1.40423 21980 | 73 |
| 0.95083 11516 | 1.09487 03382 | 0.05109 27637 | 73 38 | 1.38499 61405 | 72 |
| 0.94526 98796 | 1.09376 86463 | 0.05356 97161 | 72 43 | 1.36576 00830 | 71 |
| 0.93941 98461 | 1.09261 66042 | 0.05598 89014 | 71 48 | 1.34652 40255 | 70 |
| 0.93328 29005 | 1.09141 56156 | 0.05834 75147 | 70 52 | 1.32728 79680 | 69 |
| 0.92686 09817 | 1.09016 71440 | 0.06064 27902 | 69 56 | 1.30805 19105 | 68 |
| 0.92015 61173 | 1.08887 27107 | 0.06287 20041 | 69 1 | 1.28881 58530 | 67 |
| 0.91317 04228 | 1.08753 38930 | 0.06503 24775 | 68 5 | 1.26957 97955 | 66 |
| 0.90590 61007 | 1.08615 23221 | 0.06712 15792 | 67 9 | 1.25034 37380 | 65 |
| 0.89836 54396 | 1.08472 96815 | 0.06913 67285 | 66 12 | 1.23110 76805 | 64 |
| 0.89055 08135 | 1.08326 77048 | 0.07107 53988 | 65 16 | 1.21187 16230 | 63 |
| 0.88246 46805 | 1.08176 81732 | 0.07293 51200 | 64 19 | 1.19263 55655 | 62 |
| 0.87410 95823 | 1.08023 29140 | 0.07471 34824 | 63 23 | 1.17339 95080 | 61 |
| 0.86548 81427 | 1.07866 37978 | 0.07640 81398 | 62 26 | 1.15416 34504 | 60 |
| 0.85660 30670 | 1.07706 27365 | 0.07801 68127 | 61 29 | 1.13492 73929 | 59 |
| 0.84745 71408 | 1.07543 16809 | 0.07953 72924 | 60 31 | 1.11569 13354 | 58 |
| 0.83805 32290 | 1.07377 26184 | 0.08096 74440 | 59 34 | 1.09645 52779 | 57 |
| 0.82839 42745 | 1.07208 75705 | 0.08230 52102 | 58 36 | 1.07721 92204 | 56 |
| 0 81848 32973 | 1.07037 85902 | 0.08354 86152 | 57 39 | 1.05798 31629 | 55 |
| 0.80832 33933 | 1.06864 77599 | 0.08469 57684 | 56 41 | 1.03874 71054 | 54 |
| 0.79791 77333 | 1.06689 71884 | 0.08574 48680 | 55 43 | 1.01951 10479 | 53 |
| 0.78726 95615 | 1.06512 90086 | 0.08669 42053 | 54 44 | 1.00027 49904 | 52 |
| 0.77638 21945 | 1.06334 53750 | 0.08754 21680 | 53 46 | 0.98103 89329 | 51 |
| 0.76525 90201 | 1.06154 84606 | 0.08828 72448 | 52 48 | 0.96180 28754 | 50 |
| 0.75390 34961 | 1.05974 04548 | 0.08892 80287 | 51 49 | 0.94256 68179 | 49 |
| 0.74231 91490 | 1.05792 35605 | 0.08946 32214 | 50 49 | 0.92333 07604 | 48 |
| 0.73050 95727 | 1.05609 99913 | 0.08989 16370 | 49 50 | 0.90409 47028 | 47 |
| 0.71847 84273 | 1.05427 19690 | 0.09021 22056 | 48 50 | 0.88485 86453 | 46 |
| 0.70622 94378 | 1.05244 17208 | 0.09042 39779 | 47 51 | 0.85562 25878 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 1.7867691349, K′ = 1.9355810960, E = 1.3931402485, E′ = 1.3055390943,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.01985 29904 | 1 8 | 0.00437 25767 | 1.00004 34107 | 0.01737 52657 |
| 2 | 0.03970 59807 | 2 16 | 0.00873 86910 | 1.00017 35897 | 0.03474 53796 |
| 3 | 0.05955 89712 | 3 24 | 0.01309 18945 | 1.00039 03787 | 0.05210 51913 |
| 4 | 0.07941 19615 | 4 32 | 0.01742 57681 | 1.00069 35136 | 0.06944 95525 |
| 5 | 0.09926 49519 | 5 41 | 0.02173 39351 | 1.00108 26253 | 0.08677 33185 |
| 6 | 0.11911 79423 | 6 49 | 0.02601 00761 | 1.00155 72398 | 0.10407 13496 |
| 7 | 0.13897 09327 | 7 57 | 0.03024 79420 | 1.00211 67791 | 0.12133 85117 |
| 8 | 0.15882 39231 | 9 5 | 0.03444 13683 | 1.00276 05620 | 0.13856 96780 |
| 9 | 0.17867 69135 | 10 13 | 0.03858 42875 | 1.00348 78042 | 0.15575 97300 |
| 10 | 0.19852 99039 | 11 21 | 0.04267 07422 | 1.00429 76203 | 0.17290 35587 |
| 11 | 0.21838 28943 | 12 28 | 0.04669 48973 | 1.00518 90239 | 0.18999 60657 |
| 12 | 0.23823 58847 | 13 36 | 0.05065 10519 | 1.00616 09295 | 0.20703 21648 |
| 13 | 0.25808 88751 | 14 43 | 0.05453 36499 | 1.00721 21534 | 0.22400 67828 |
| 14 | 0.27794 18655 | 15 51 | 0.05833 72913 | 1.00834 14154 | 0.24091 48609 |
| 15 | 0.29779 48558 | 16 58 | 0.06205 67422 | 1.00954 73402 | 0.25775 13559 |
| 16 | 0.31764 78462 | 18 5 | 0.06568 69435 | 1.01082 84592 | 0.27451 12417 |
| 17 | 0.33750 08366 | 19 12 | 0.06922 30203 | 1.01218 32120 | 0.29118 95099 |
| 18 | 0.35735 38270 | 20 18 | 0.07266 02895 | 1.01360 99487 | 0.30778 11718 |
| 19 | 0.37720 68174 | 21 25 | 0.07599 42673 | 1.01510 69318 | 0.32428 12593 |
| 20 | 0.39705 98078 | 22 31 | 0.07922 06754 | 1.01667 23379 | 0.34068 48260 |
| 21 | 0.41691 27981 | 23 37 | 0.08233 54475 | 1.01830 42606 | 0.35698 69491 |
| 22 | 0.43676 57885 | 24 42 | 0.08533 47336 | 1.02000 07123 | 0.37318 27300 |
| 23 | 0.45661 87789 | 25 48 | 0.08821 49046 | 1.02175 96267 | 0.38926 72959 |
| 24 | 0.47647 17693 | 26 53 | 0.09097 25564 | 1.02357 88616 | 0.40523 58014 |
| 25 | 0.49632 47597 | 27 59 | 0.09360 45123 | 1.02545 62012 | 0.42108 34293 |
| 26 | 0.51617 77501 | 29 4 | 0.09610 78252 | 1.02738 93589 | 0.43680 53924 |
| 27 | 0.53603 07405 | 30 8 | 0.09847 97792 | 1.02937 59801 | 0.45239 69344 |
| 28 | 0.55588 37309 | 31 13 | 0.10071 78905 | 1.03141 36450 | 0.46785 33318 |
| 29 | 0.57573 67212 | 32 17 | 0.10281 99075 | 1.03349 98717 | 0.48316 98948 |
| 30 | 0.59558 97116 | 33 22 | 0.10478 38101 | 1.03563 21191 | 0.49834 19688 |
| 31 | 0.61544 27020 | 34 25 | 0.10660 78092 | 1.03780 77899 | 0.51336 49360 |
| 32 | 0.63529 56924 | 35 28 | 0.10829 03444 | 1.04002 42340 | 0.52823 42166 |
| 33 | 0.65514 86828 | 36 31 | 0.10983 00821 | 1.04227 87515 | 0.54294 52702 |
| 34 | 0.67500 16732 | 37 34 | 0.11122 59132 | 1.04456 85961 | 0.55749 35973 |
| 35 | 0.69485 46636 | 38 37 | 0.11247 69491 | 1.04689 09786 | 0.57187 47405 |
| 36 | 0.71470 76540 | 39 39 | 0.11358 25187 | 1.04924 30699 | 0.58608 42864 |
| 37 | 0.73456 06443 | 40 41 | 0.11454 21645 | 1.05162 20047 | 0.60011 78665 |
| 38 | 0.75441 36347 | 41 42 | 0.11535 56375 | 1.05402 48851 | 0.61397 11590 |
| 39 | 0.77426 66251 | 42 44 | 0.11602 28932 | 1.05644 87839 | 0.62763 98902 |
| 40 | 0.79411 96155 | 43 46 | 0.11654 40861 | 1.05889 07481 | 0.64111 98356 |
| 41 | 0.81397 26059 | 44 46 | 0.11691 95649 | 1.06134 78029 | 0.65440 68220 |
| 42 | 0.83382 55963 | 45 47 | 0.11714 98662 | 1.06381 69550 | 0.66749 67282 |
| 43 | 0.85367 85867 | 46 47 | 0.11723 57096 | 1.06629 51962 | 0.68038 54871 |
| 44 | 0.87353 15771 | 47 48 | 0.11717 79914 | 1.06877 95074 | 0.69306 90869 |
| 45 | 0.89338 45674 | 48 48 | 0.11697 77784 | 1.07126 68617 | 0.70554 35725 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 40°$

$q = 0.033265256695577$, $\Theta 0 = 0.9334719356$, $HK = 0.8550825245$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.14254 42177 | 0.00000 00000 | 90° 0′ | 1.78676 91349 | 90 |
| 0.99984 63487 | 1.14250 07942 | 0.00382 84907 | 89 8 | 1.76691 61445 | 89 |
| 0.99938 54451 | 1.14237 05769 | 0.00765 31872 | 88 15 | 1.74706 31541 | 88 |
| 0.99861 74408 | 1.14215 37243 | 0.01147 02963 | 87 23 | 1.72721 01637 | 87 |
| 0.99754 25881 | 1.14185 05008 | 0.01527 60269 | 86 30 | 1.70735 71733 | 86 |
| 0.99616 12401 | 1.14146 12760 | 0.01906 65913 | 85 38 | 1.68750 41829 | 85 |
| 0.99447 38506 | 1.14098 65243 | 0.02283 82057 | 84 46 | 1.66765 11926 | 84 |
| 0.99248 09734 | 1.14042 68243 | 0.02658 70918 | 83 53 | 1.64779 82022 | 83 |
| 0.99018 32628 | 1.13978 28584 | 0.03030 94781 | 83 1 | 1.62794 52118 | 82 |
| 0.98758 14726 | 1.13905 54113 | 0.03400 16009 | 82 8 | 1.60809 22214 | 81 |
| 0.98467 64560 | 1.13824 53698 | 0.03765 97054 | 81 16 | 1.58823 92310 | 80 |
| 0.98146 91652 | 1.13735 37211 | 0.04128 00477 | 80 23 | 1.56838 62406 | 79 |
| 0.97796 06509 | 1.13638 15521 | 0.04485 88958 | 79 30 | 1.54853 32502 | 78 |
| 0.97415 20616 | 1.13533 00476 | 0.04839 25314 | 78 37 | 1.52868 02598 | 77 |
| 0.97004 46432 | 1.13420 04893 | 0.05187 72514 | 77 44 | 1.50882 72694 | 76 |
| 0.96563 97386 | 1.13299 42539 | 0.05530 93702 | 76 51 | 1.48897 42791 | 75 |
| 0.96093 87866 | 1.13171 28116 | 0.05868 52206 | 75 57 | 1.46912 12887 | 74 |
| 0.95594 33213 | 1.13035 77242 | 0.06200 11573 | 75 4 | 1.44926 82983 | 73 |
| 0.95065 49716 | 1.12893 06433 | 0.06525 35577 | 74 10 | 1.42941 53079 | 72 |
| 0.94507 54604 | 1.12743 33082 | 0.06843 88251 | 73 17 | 1.40956 23175 | 71 |
| 0.93920 66032 | 1.12586 75438 | 0.07155 33910 | 72 23 | 1.38970 93271 | 70 |
| 0.93305 03082 | 1.12423 52584 | 0.07459 37177 | 71 29 | 1.36985 63367 | 69 |
| 0.92660 85744 | 1.12253 84414 | 0.07755 63011 | 70 34 | 1.35000 33463 | 68 |
| 0.91988 34913 | 1.12077 91607 | 0.08043 76736 | 69 40 | 1.33015 03560 | 67 |
| 0.91287 72377 | 1.11895 95604 | 0.08323 44077 | 68 45 | 1.31029 73656 | 66 |
| 0.90559 20807 | 1.11708 18582 | 0.08594 31188 | 67 51 | 1.29044 43752 | 65 |
| 0.89803 03745 | 1.11514 83422 | 0.08856 04692 | 66 56 | 1.27059 13848 | 64 |
| 0.89019 45598 | 1.11316 13690 | 0.09108 31714 | 66 0 | 1.25073 83944 | 63 |
| 0.88208 71618 | 1.11112 33599 | 0.09350 79923 | 65 5 | 1.23088 54040 | 62 |
| 0.87371 07901 | 1.10903 67986 | 0.09583 17573 | 64 9 | 1.21103 24136 | 61 |
| 0.86506 81367 | 1.10690 42279 | 0.09805 13545 | 63 14 | 1.19117 94233 | 60 |
| 0.85616 19751 | 1.10472 82465 | 0.10016 37391 | 62 18 | 1.17132 64329 | 59 |
| 0.84699 51593 | 1.10251 15061 | 0.10216 59383 | 61 21 | 1.15147 34425 | 58 |
| 0.83757 06220 | 1.10025 67080 | 0.10405 50557 | 60 25 | 1.13162 04521 | 57 |
| 0.82789 13739 | 1.09796 65999 | 0.10582 82770 | 59 28 | 1.11176 74617 | 56 |
| 0.81796 05020 | 1.09564 39724 | 0.10748 28746 | 58 32 | 1.09191 44713 | 55 |
| 0.80778 11684 | 1.09329 16556 | 0.10901 62132 | 57 34 | 1.07206 14809 | 54 |
| 0.79735 66091 | 1.09091 25160 | 0.11042 57553 | 56 37 | 1.05220 84905 | 53 |
| 0.78669 01322 | 1.08850 94525 | 0.11170 90668 | 55 39 | 1.03235 55001 | 52 |
| 0.77578 51173 | 1.08608 53932 | 0.11286 38228 | 54 42 | 1.01250 25098 | 51 |
| 0.76464 50133 | 1.08364 32917 | 0.11388 78137 | 53 44 | 0.99264 95194 | 50 |
| 0.75327 33376 | 1.08118 61237 | 0.11477 89511 | 52 45 | 0.97279 65290 | 49 |
| 0.74167 36742 | 1.07871 68830 | 0.11553 52736 | 51 46 | 0.95294 35386 | 48 |
| 0.72984 96728 | 1.07623 85782 | 0.11615 49535 | 50 46 | 0.93309 05482 | 47 |
| 0.71780 50468 | 1.07375 42288 | 0.11663 63025 | 49 47 | 0.91323 75578 | 46 |
| 0.70554 35725 | 1.07126 68617 | 0.11697 77784 | 48 48 | 0.89338 45674 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = K' = 1.8540746773, \quad E = E' = 1.3506438810,$

| r | $F\phi$ | $\phi$ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.02060 08297 | 1 11 | 0.00559 22185 | 1.00005 76114 | 0.01732 23240 |
| 2 | 0.04120 16595 | 2 22 | 0.01117 56998 | 1.00023 03752 | 0.03463 96092 |
| 3 | 0.06180 24892 | 3 32 | 0.01674 17286 | 1.00051 80814 | 0.05194 68175 |
| 4 | 0.08240 33190 | 4 43 | 0 02228 16343 | 1.00092 03796 | 0.06923 89126 |
| 5 | 0.10300 41487 | 5 54 | 0.02778 68124 | 1.00143 67802 | 0.08651 08611 |
| 6 | 0.12360 49785 | 7 4 | 0.03324 87460 | 1.00206 66547 | 0.10375 76329 |
| 7 | 0.14420 58082 | 8 15 | 0.03865 90273 | 1.00280 92364 | 0.12097 42023 |
| 8 | 0.16480 66380 | 9 25 | 0.04400 93780 | 1.00366 36213 | 0.13815 55494 |
| 9 | 0.18540 74677 | 10 36 | 0.04929 16689 | 1.00462 87696 | 0.15529 66598 |
| 10 | 0.20600 82975 | 11 46 | 0.05449 79400 | 1.00570 35065 | 0.17239 25270 |
| 11 | 0.22660 91272 | 12 56 | 0.05962 04166 | 1.00688 65237 | 0.18943 81524 |
| 12 | 0.24720 99570 | 14 6 | 0.06465 15306 | 1.00817 63813 | 0.20642 85463 |
| 13 | 0.26781 07867 | 15 15 | 0.06958 39334 | 1.00957 15091 | 0.22335 87294 |
| 14 | 0.28841 16165 | 16 25 | 0.07441 05129 | 1.01107 02088 | 0.24022 37330 |
| 15 | 0.30901 24462 | 17 34 | 0.07912 44078 | 1.01267 06562 | 0.25701 86008 |
| 16 | 0.32961 32760 | 18 43 | 0.08371 90207 | 1.01437 09030 | 0.27373 83893 |
| 17 | 0.35021 41057 | 19 52 | 0.08818 80301 | 1.01616 88793 | 0.29037 81691 |
| 18 | 0.37081 49355 | 21 1 | 0.09252 54012 | 1.01806 23965 | 0.30693 30262 |
| 19 | 0.39141 57652 | 22 9 | 0.09672 53955 | 1.02004 91494 | 0.32339 80622 |
| 20 | 0.41201 65950 | 23 17 | 0.10078 25794 | 1.02212 67193 | 0.33976 83967 |
| 21 | 0.43261 74247 | 24 25 | 0.10469 18308 | 1.02429 25769 | 0.35603 91671 |
| 22 | 0.45321 82545 | 25 33 | 0.10844 83455 | 1.02654 40853 | 0.37220 55308 |
| 23 | 0.47381 90842 | 26 40 | 0.11204 76417 | 1.02887 85035 | 0.38826 26656 |
| 24 | 0.49441 99139 | 27 47 | 0.11548 55630 | 1.03129 29893 | 0.40420 57714 |
| 25 | 0.51502 07437 | 28 54 | 0.11875 82813 | 1.03378 46028 | 0.42003 00711 |
| 26 | 0.53562 15734 | 30 0 | 0.12186 22978 | 1.03635 03103 | 0.43573 08120 |
| 27 | 0.55622 24032 | 31 6 | 0.12479 44425 | 1.03898 69880 | 0.45130 32670 |
| 28 | 0.57682 32329 | 32 12 | 0.12755 18736 | 1.04169 14251 | 0.46674 27359 |
| 29 | 0.59742 40627 | 33 17 | 0.13013 20757 | 1.04446 03288 | 0.48204 45468 |
| 30 | 0.61802 48924 | 34 22 | 0.13253 28561 | 1.04729 03271 | 0.49720 40572 |
| 31 | 0.63862 57222 | 35 27 | 0.13475 23413 | 1.05017 79739 | 0.51221 66556 |
| 32 | 0.65922 65519 | 36 32 | 0.13678 89725 | 1.05311 97528 | 0.52707 77628 |
| 33 | 0.67982 73817 | 37 36 | 0.13864 14993 | 1.05611 20812 | 0.54178 28334 |
| 34 | 0.70042 82114 | 38 39 | 0.14030 89744 | 1.05915 13149 | 0.55632 73569 |
| 35 | 0.72102 90412 | 39 43 | 0.14179 07457 | 1.06223 37524 | 0.57070 68597 |
| 36 | 0.74162 98709 | 40 46 | 0.14308 64509 | 1.06535 56397 | 0.58491 69061 |
| 37 | 0.76223 07007 | 41 48 | 0.14419 60059 | 1.06851 31742 | 0.59895 31001 |
| 38 | 0.78283 15304 | 42 51 | 0.14511 96000 | 1.07170 25103 | 0.61281 10868 |
| 39 | 0.80343 23602 | 43 54 | 0.14585 76849 | 1.07491 97630 | 0.62648 65539 |
| 40 | 0.82403 31899 | 44 54 | 0.14641 09671 | 1.07816 10137 | 0.63997 52334 |
| 41 | 0.84463 40197 | 45 55 | 0.14678 03964 | 1.08142 23139 | 0.65327 29030 |
| 42 | 0.86523 48494 | 46 56 | 0.14696 71583 | 1.08469 96910 | 0.66637 53880 |
| 43 | 0.88583 56792 | 47 57 | 0.14697 26631 | 1.08798 91523 | 0.67927 85625 |
| 44 | 0.90643 65089 | 48 57 | 0.14679 85365 | 1.09128 66907 | 0.69197 83514 |
| 45 | 0 92703 73387 | 49 57 | 0.14644 66094 | 1.09458 82886 | 0.70447 07318 |
| 90 - r | $F\psi$ | $\psi$ | G(r). | C(r) | B(r) |

TABLE $\theta = 45°$

$q = e^{-\pi} = 0.04321391826377$, $\Theta 0 = 0.9135791382$, $HK = 0.9135791382$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.18920 71150 | 0.00000 00000 | 90° 0′ | 1.85407 46773 | 90 |
| 0.99984 54246 | 1.18914 94665 | 0.00470 60108 | 89 10 | 1.83347 38476 | 89 |
| 0.99938 17514 | 1.18897 65912 | 0.00940 76502 | 88 20 | 1.81287 30178 | 88 |
| 0.99860 91406 | 1.18868 87000 | 0.01410 05467 | 87 30 | 1.79227 21881 | 87 |
| 0.99752 78584 | 1.18828 61440 | 0.01878 03289 | 86 40 | 1.77167 13583 | 86 |
| 0.99613 82775 | 1.18776 94140 | 0.02344 26255 | 85 49 | 1.75107 05286 | 85 |
| 0.99444 08767 | 1.18713 91403 | 0.02808 30653 | 84 59 | 1.73046 96988 | 84 |
| 0.99243 62407 | 1.18639 60914 | 0.03269 72774 | 84 9 | 1.70986 88691 | 83 |
| 0.99012 50593 | 1.18554 11736 | 0.03728 08916 | 83 18 | 1.68926 80393 | 82 |
| 0.98750 81276 | 1.18457 54293 | 0.04182 95382 | 82 28 | 1.66866 72096 | 81 |
| 0.98458 63450 | 1.18350 00363 | 0.04633 88487 | 81 37 | 1.64806 63798 | 80 |
| 0.98136 07151 | 1.18231 63059 | 0.05080 44575 | 80 47 | 1.62746 55501 | 79 |
| 0.97783 23446 | 1.18102 56817 | 0.05522 19994 | 79 56 | 1.60686 47203 | 78 |
| 0.97400 24430 | 1.17962 97376 | 0.05958 71139 | 79 5 | 1.58626 38906 | 77 |
| 0.96987 23216 | 1.17813 01756 | 0.06389 54439 | 78 14 | 1.56566 30608 | 76 |
| 0.96544 33929 | 1.17652 88244 | 0.06814 26379 | 77 23 | 1.54506 22311 | 75 |
| 0.96071 71696 | 1.17482 76366 | 0.07232 43506 | 76 32 | 1.52446 14013 | 74 |
| 0.95569 52639 | 1.17302 86866 | 0.07643 62449 | 75 40 | 1.50386 05716 | 73 |
| 0.95037 93863 | 1.17113 41680 | 0.08047 39933 | 74 48 | 1.48325 97418 | 72 |
| 0.94477 13447 | 1.16914 63907 | 0.08443 32799 | 73 57 | 1.46265 89121 | 71 |
| 0.93887 30433 | 1.16706 77783 | 0.08830 98027 | 73 5 | 1.44205 80823 | 70 |
| 0.93268 64814 | 1.16490 08653 | 0.09209 92756 | 72 13 | 1.42145 72526 | 69 |
| 0.92621 37526 | 1.16264 82937 | 0.09579 74315 | 71 20 | 1.40085 64228 | 68 |
| 0.91945 70430 | 1.16031 28097 | 0.09940 00252 | 70 27 | 1.38025 55931 | 67 |
| 0.91241 86305 | 1.15789 72608 | 0.10290 28362 | 69 34 | 1.35965 47634 | 66 |
| 0.90510 08831 | 1.15540 45920 | 0.10630 16727 | 68 41 | 1.33905 39336 | 65 |
| 0.89750 62579 | 1.15283 78419 | 0.10959 23752 | 67 48 | 1.31845 31039 | 64 |
| 0.88963 72995 | 1.15020 01398 | 0.11277 08206 | 66 54 | 1.29785 22741 | 63 |
| 0.88149 66386 | 1.14749 47011 | 0.11583 29266 | 66 0 | 1.27725 14444 | 62 |
| 0.87308 69906 | 1.14472 48239 | 0.11877 46567 | 65 6 | 1.25665 06146 | 61 |
| 0.86441 11542 | 1.14189 38846 | 0.12159 20252 | 64 11 | 1.23604 97849 | 60 |
| 0.85547 20099 | 1.13900 53339 | 0.12428 11025 | 63 16 | 1.21544 89551 | 59 |
| 0.84627 25182 | 1.13606 26928 | 0.12683 80211 | 62 21 | 1.19484 81254 | 58 |
| 0.83681 57184 | 1.13306 95480 | 0.12925 89815 | 61 26 | 1.17424 72956 | 57 |
| 0.82710 47269 | 1.13002 95477 | 0.13154 02588 | 60 30 | 1.15364 64659 | 56 |
| 0.81714 27355 | 1.12694 63970 | 0.13367 82099 | 59 34 | 1.13304 56361 | 55 |
| 0.80693 30099 | 1.12382 38537 | 0.13566 92789 | 58 38 | 1.11244 48064 | 54 |
| 0.79647 88881 | 1.12066 57231 | 0.13751 00077 | 57 42 | 1.09184 39766 | 53 |
| 0.78578 37785 | 1.11747 58542 | 0.13919 70407 | 56 45 | 1.07124 31469 | 52 |
| 0.77485 11587 | 1.11425 81342 | 0.14072 71344 | 55 47 | 1.05064 23171 | 51 |
| 0.76368 45735 | 1.11101 64844 | 0.14209 71663 | 54 50 | 1.03004 14874 | 50 |
| 0.75228 76332 | 1.10775 48548 | 0.14330 41415 | 53 52 | 1.00944 06576 | 49 |
| 0.74066 40121 | 1.10447 72199 | 0.14434 52037 | 52 53 | 0.98883 98279 | 48 |
| 0.72881 74469 | 1.10118 75735 | 0.14521 76436 | 51 55 | 0.96823 89981 | 47 |
| 0.71675 17348 | 1.09788 99237 | 0.14591 89078 | 50 56 | 0.94763 81684 | 46 |
| 0.70447 07318 | 1.09458 82886 | 0.14644 66094 | 49 57 | 0.92703 73387 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 1.9355810960, K′ = 1.7867691349, E = 1.3055390943, E′ = 1.3931402485,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.02150 64566 | 1 14 | 0.00699 85212 | 1.00007 52700 | 0.01724 17831 |
| 2 | 0.04301 29132 | 2 28 | 0.01398 53763 | 1.00030 09884 | 0.03447 86990 |
| 3 | 0.06451 93699 | 3 41 | 0.02094 89334 | 1.00067 68809 | 0.05170 58810 |
| 4 | 0.08602 58265 | 4 55 | 0.02787 76288 | 1.00120 24903 | 0.06891 84630 |
| 5 | 0.10753 22831 | 6 9 | 0.03476 00006 | 1.00187 71775 | 0.08611 15805 |
| 6 | 0.12903 87397 | 7 22 | 0.04158 42717 | 1.00270 01222 | 0.10328 03705 |
| 7 | 0.15054 51963 | 8 36 | 0.04834 06320 | 1.00367 03237 | 0.12041 99725 |
| 8 | 0.17205 16530 | 9 49 | 0.05501 67694 | 1.00478 66023 | 0.13752 55283 |
| 9 | 0.19355 81096 | 11 3 | 0.06160 24003 | 1.00604 76005 | 0.15459 21831 |
| 10 | 0.21506 45662 | 12 16 | 0.06808 70479 | 1.00745 17850 | 0.17161 50856 |
| 11 | 0.23657 10228 | 13 28 | 0.07446 05194 | 1.00899 74482 | 0.18858 93888 |
| 12 | 0.25807 74795 | 14 41 | 0.08071 29320 | 1.01068 27105 | 0.20551 02505 |
| 13 | 0.27958 39361 | 15 53 | 0.08683 47367 | 1.01250 55225 | 0.22237 28335 |
| 14 | 0.30109 03927 | 17 6 | 0.09281 67403 | 1.01446 36673 | 0.23917 23067 |
| 15 | 0.32259 68493 | 18 18 | 0.09865 01256 | 1.01655 47635 | 0.25590 38457 |
| 16 | 0.34410 33059 | 19 29 | 0.10432 64694 | 1.01877 62678 | 0.27256 26330 |
| 17 | 0.36560 97626 | 20 40 | 0.10983 77593 | 1.02112 54784 | 0.28914 38591 |
| 18 | 0.38711 62192 | 21 51 | 0.11517 64068 | 1.02359 95379 | 0.30564 27234 |
| 19 | 0.40862 26758 | 23 2 | 0.12033 52604 | 1.02619 54370 | 0.32205 44344 |
| 20 | 0.43012 91324 | 24 13 | 0.12530 76146 | 1.02891 00179 | 0.33837 42110 |
| 21 | 0.45163 55891 | 25 22 | 0.13008 72182 | 1.03173 99787 | 0.35459 72832 |
| 22 | 0.47314 20457 | 26 31 | 0.13466 82799 | 1.03468 18764 | 0.37071 88930 |
| 23 | 0.49464 85023 | 27 41 | 0.13904 54724 | 1.03773 21323 | 0.38673 42953 |
| 24 | 0.51615 49589 | 28 50 | 0.14321 39340 | 1.04088 70352 | 0.40263 87589 |
| 25 | 0.53766 14155 | 29 59 | 0.14716 92687 | 1.04414 27466 | 0.41842 75678 |
| 26 | 0.55916 78722 | 31 6 | 0.15090 75443 | 1.04749 53052 | 0.43409 60218 |
| 27 | 0.58067 43288 | 32 14 | 0.15442 52892 | 1.05094 06315 | 0.44963 94381 |
| 28 | 0.60218 07854 | 33 21 | 0.15771 94871 | 1.05447 45329 | 0.46505 31522 |
| 29 | 0.62368 72420 | 34 29 | 0.16078 75703 | 1.05809 27090 | 0.48033 25191 |
| 30 | 0.64519 36987 | 35 36 | 0.16362 74123 | 1.06179 07561 | 0.49547 29148 |
| 31 | 0.66670 01553 | 36 41 | 0.16623 73178 | 1.06556 41737 | 0.51046 97376 |
| 32 | 0.68820 66119 | 37 46 | 0.16861 60131 | 1.06940 83686 | 0.52531 84091 |
| 33 | 0.70971 30685 | 38 51 | 0.17076 26341 | 1.07331 86617 | 0.54001 43761 |
| 34 | 0.73121 95251 | 39 56 | 0.17267 67142 | 1.07729 02929 | 0.55455 31119 |
| 35 | 0.75272 59818 | 41 1 | 0.17435 81713 | 1.08131 84270 | 0.56893 01177 |
| 36 | 0.77423 24384 | 42 4 | 0.17580 72936 | 1.08539 81601 | 0.58314 09242 |
| 37 | 0.79573 88950 | 43 7 | 0.17702 47258 | 1.08952 45247 | 0.59718 10935 |
| 38 | 0.81724 53516 | 44 9 | 0.17801 14536 | 1.09369 24965 | 0.61104 62201 |
| 39 | 0.83875 18083 | 45 12 | 0.17876 87890 | 1.09789 70001 | 0.62473 19335 |
| 40 | 0.86025 82649 | 46 15 | 0.17929 83544 | 1.10213 29153 | 0.63823 38991 |
| 41 | 0.88176 47215 | 47 15 | 0.17960 20675 | 1.10639 50831 | 0.65154 78204 |
| 42 | 0.90327 11781 | 48 16 | 0.17968 21252 | 1.11067 83124 | 0.66466 94406 |
| 43 | 0.92477 76347 | 49 16 | 0.17954 09878 | 1.11497 73861 | 0.67759 45449 |
| 44 | 0.94628 40914 | 50 17 | 0.17918 13641 | 1.11928 70673 | 0.69031 89618 |
| 45 | 0.96779 05480 | 51 17 | 0.17860 61952 | 1.12360 21058 | 0.70283 85652 |
| 90−r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 50°$

$q = 0.055019933698829$, $\Theta 0 = 0.8899784604$, $HK = 0.9715669451$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90-r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.24728 65857 | 0.00000 00000 | 90° 0′ | 1.93558 10960 | 90 |
| 0.99984 40186 | 1.24721 12154 | 0.00561 92362 | 89 12 | 1.91407 46394 | 89 |
| 0.99937 61319 | 1.24698 51964 | 0.01123 36482 | 88 25 | 1.89256 81828 | 88 |
| 0.99859 65127 | 1.24660 88048 | 0.01683 84106 | 87 37 | 1.87106 17261 | 87 |
| 0.99750 54487 | 1.24608 24999 | 0.02242 89646 | 86 50 | 1.84955 52695 | 86 |
| 0.99610 33424 | 1.24540 69243 | 0.02799 96670 | 86 2 | 1.82804 88129 | 85 |
| 0.99439 07108 | 1.24458 29027 | 0.03354 64884 | 85 14 | 1.80654 23563 | 84 |
| 0.99236 81849 | 1.24361 14410 | 0.03906 43123 | 84 26 | 1.78503 58997 | 83 |
| 0.99003 65093 | 1.24249 37250 | 0.04454 82835 | 83 39 | 1.76352 94430 | 82 |
| 0.98739 65416 | 1.24123 11192 | 0.04999 35367 | 82 51 | 1.74202 29864 | 81 |
| 0.98444 92517 | 1.23982 51648 | 0.05539 51961 | 82 3 | 1.72051 65298 | 80 |
| 0.98119 57210 | 1.23827 75779 | 0.06074 83740 | 81 14 | 1.69901 00732 | 79 |
| 0.97763 71417 | 1.23659 02476 | 0.06604 81700 | 80 26 | 1.67750 36165 | 78 |
| 0.97377 48160 | 1.23476 52334 | 0.07128 96708 | 79 37 | 1.65599 71599 | 77 |
| 0.96961 01546 | 1.23280 47629 | 0.07646 79497 | 78 49 | 1.63449 07033 | 76 |
| 0.96514 46762 | 1.23071 12287 | 0.08157 80662 | 78 0 | 1.61298 42467 | 75 |
| 0.96038 00059 | 1.22848 71860 | 0.08661 50665 | 77 10 | 1.59147 77901 | 74 |
| 0.95531 78745 | 1.22613 53491 | 0.09157 39836 | 76 21 | 1.56997 13334 | 73 |
| 0.94996 01167 | 1.22365 85882 | 0.09644 98379 | 75 31 | 1.54846 48768 | 72 |
| 0.94430 86698 | 1.22105 99257 | 0.10123 76383 | 74 42 | 1.52695 84202 | 71 |
| 0.93836 55727 | 1.21834 25328 | 0.10593 23833 | 73 52 | 1.50545 19636 | 70 |
| 0.93213 29639 | 1.21550 97252 | 0.11052 90627 | 73 1 | 1.48394 55069 | 69 |
| 0.92561 30802 | 1.21256 49596 | 0.11502 26595 | 72 11 | 1.46243 90503 | 68 |
| 0.91880 82552 | 1.20951 18289 | 0.11940 81521 | 71 20 | 1.44093 25937 | 67 |
| 0.91172 09173 | 1.20635 40582 | 0.12368 05174 | 70 30 | 1.41942 61371 | 66 |
| 0.90435 35883 | 1.20309 54999 | 0.12783 47335 | 69 39 | 1.39791 96805 | 65 |
| 0.89670 88815 | 1.19974 01294 | 0.13186 57834 | 68 47 | 1.37641 32238 | 64 |
| 0.88878 94998 | 1.19629 20396 | 0.13576 86595 | 67 55 | 1.35490 67672 | 63 |
| 0.88059 82341 | 1.19275 54368 | 0.13953 83674 | 67 2 | 1 33340 03106 | 62 |
| 0.87213 79612 | 1.18913 46345 | 0.14316 99314 | 66 10 | 1.31189 38540 | 61 |
| 0.86341 16420 | 1.18543 40490 | 0.14665 83999 | 65 18 | 1.29038 73973 | 60 |
| 0.85442 23195 | 1.18165 81935 | 0.14999 88516 | 64 24 | 1.26888 09407 | 59 |
| 0.84517 31166 | 1.17781 16727 | 0.15318 64017 | 63 30 | 1.24737 44841 | 58 |
| 0.83566 72345 | 1.17389 91774 | 0.15621 62095 | 62 36 | 1.22586 80275 | 57 |
| 0.82590 79506 | 1.16992 54783 | 0.15908 34859 | 61 42 | 1.20436 15709 | 56 |
| 0.81589 86161 | 1.16589 54205 | 0.16178 35017 | 60 48 | 1.18285 51142 | 55 |
| 0.80564 26543 | 1.16181 39175 | 0.16431 15964 | 59 52 | 1.16134 86576 | 54 |
| 0.79514 35583 | 1.15768 59453 | 0.16666 31878 | 58 56 | 1.13984 22010 | 53 |
| 0.78440 48891 | 1.15351 65361 | 0.16883 37818 | 58 0 | 1.11833 57444 | 52 |
| 0.77343 02735 | 1.14931 07723 | 0.17081 89832 | 57 4 | 1.09682 92877 | 51 |
| 0.76222 34019 | 1.14507 37802 | 0.17261 45069 | 56 8 | 1.07532 28311 | 50 |
| 0.75078 80264 | 1.14081 07240 | 0.17421 61892 | 55 10 | 1.05381 63745 | 49 |
| 0.73912 79584 | 1.13652 67992 | 0.17562 00006 | 54 12 | 1.03230 99179 | 48 |
| 0.72724 70671 | 1.13222 72263 | 0.17682 20583 | 53 13 | 1.01080 34613 | 47 |
| 0.71514 92767 | 1.12791 72446 | 0.17781 86395 | 52 15 | 0.98929 70046 | 46 |
| 0.70283 85652 | 1.12360 21058 | 0.17860 61952 | 51 17 | 0.96779 05480 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = 2.0347153122$, $K' = 1.7312451757$, $E = 1.2586796248$, $E' = 1.4322909693$,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.02260 79479 | 1 18 | 0.00862 00346 | 1.00009 74600 | 0.01712 13223 |
| 2 | 0.04521 58958 | 2 35 | 0.01722 45749 | 1.00038 97217 | 0.03423 80342 |
| 3 | 0.06782 38437 | 3 53 | 0.02579 81795 | 1.00087 64305 | 0.05134 55249 |
| 4 | 0.09043 17916 | 5 10 | 0.03432 55123 | 1.00155 69957 | 0.06843 91832 |
| 5 | 0.11303 97395 | 6 28 | 0.04279 13942 | 1.00243 05914 | 0.08551 43971 |
| 6 | 0.13564 76875 | 7 45 | 0.05118 08539 | 1.00349 61575 | 0.10256 65538 |
| 7 | 0.15825 56354 | 9 2 | 0.05947 91769 | 1.00475 24006 | 0.11959 10390 |
| 8 | 0.18086 35833 | 10 19 | 0.06767 19530 | 1.00619 77962 | 0.13658 32373 |
| 9 | 0.20347 15312 | 11 36 | 0.07574 51216 | 1.00783 05901 | 0.15353 85318 |
| 10 | 0.22607 94791 | 12 52 | 0.08368 50144 | 1.00964 88003 | 0.17045 23039 |
| 11 | 0.24868 74270 | 14 9 | 0.09147 83960 | 1.01165 02201 | 0.18731 99332 |
| 12 | 0.27129 53749 | 15 25 | 0.09911 25013 | 1.01383 24199 | 0.20413 67975 |
| 13 | 0.29390 33229 | 16 40 | 0.10657 50694 | 1.01619 27508 | 0.22089 82730 |
| 14 | 0.31651 12708 | 17 56 | 0.11385 43755 | 1.01872 83473 | 0.23759 97340 |
| 15 | 0.33911 92187 | 19 11 | 0.12093 92580 | 1.02143 61311 | 0.25423 65532 |
| 16 | 0.36172 71666 | 20 25 | 0.12781 91435 | 1.02431 28147 | 0.27080 41017 |
| 17 | 0.38433 51145 | 21 40 | 0.13448 40670 | 1.02735 49050 | 0.28729 77496 |
| 18 | 0.40694 30624 | 22 54 | 0.14092 46901 | 1.03055 87080 | 0.30371 28656 |
| 19 | 0.42955 10103 | 24 7 | 0.14713 23140 | 1.03392 03331 | 0.32004 48178 |
| 20 | 0.45215 89583 | 25 20 | 0.15309 88906 | 1.03743 56974 | 0.33628 89743 |
| 21 | 0.47476 69062 | 26 33 | 0.15881 70288 | 1.04110 05314 | 0.35244 07031 |
| 22 | 0.49737 48541 | 27 45 | 0.16427 99989 | 1.04491 03831 | 0.36849 53729 |
| 23 | 0.51998 28020 | 28 56 | 0.16948 17327 | 1.04886 06244 | 0.38444 83538 |
| 24 | 0.54259 07499 | 30 8 | 0.17441 68208 | 1.05294 64558 | 0.40029 50181 |
| 25 | 0.56519 86978 | 31 18 | 0.17908 05075 | 1.05716 29130 | 0.41603 07408 |
| 26 | 0.58780 66457 | 32 28 | 0.18346 86827 | 1.06150 48720 | 0.43165 09003 |
| 27 | 0.61041 45937 | 33 38 | 0.18757 78710 | 1.06596 70560 | 0.44715 08801 |
| 28 | 0.63302 25418 | 34 46 | 0.19140 52188 | 1.07054 40415 | 0.46253 60691 |
| 29 | 0.65563 04895 | 35 55 | 0.19494 84794 | 1.07523 02647 | 0.47777 18627 |
| 30 | 0.67823 84374 | 37 3 | 0.19820 59959 | 1.08002 00285 | 0.49288 36645 |
| 31 | 0.70084 63853 | 38 10 | 0.20117 66827 | 1.08490 75092 | 0.50785 68872 |
| 32 | 0.72345 43332 | 39 16 | 0.20386 00053 | 1.08988 67634 | 0.52268 69541 |
| 33 | 0.74606 22811 | 40 23 | 0.20625 59591 | 1.09495 17358 | 0.53736 93004 |
| 34 | 0.76867 02290 | 41 28 | 0.20836 50468 | 1.10009 62656 | 0.55189 93747 |
| 35 | 0.79127 81769 | 42 33 | 0.21018 82554 | 1.10531 40947 | 0.56627 26408 |
| 36 | 0.81388 61249 | 43 38 | 0.21172 70324 | 1.11059 88749 | 0.58048 45794 |
| 37 | 0.83649 40728 | 44 41 | 0.21298 32611 | 1.11594 41760 | 0.59453 06894 |
| 38 | 0.85910 20207 | 45 45 | 0.21395 92364 | 1.12134 34929 | 0.60840 64905 |
| 39 | 0.88170 99686 | 46 48 | 0.21465 76400 | 1.12679 02542 | 0.62210 75244 |
| 40 | 0.90431 79165 | 47 50 | 0.21508 15155 | 1.13227 78297 | 0.63562 93571 |
| 41 | 0.92692 58644 | 48 51 | 0.21523 42440 | 1.13779 95386 | 0.64896 75812 |
| 42 | 0.94953 38123 | 49 53 | 0.21511 95200 | 1.14334 86579 | 0.66211 78175 |
| 43 | 0.97214 17602 | 50 53 | 0.21474 13276 | 1.14891 84299 | 0.67507 57177 |
| 44 | 0.99474 97081 | 51 53 | 0.21410 39170 | 1.15450 20711 | 0.68783 69663 |
| 45 | 1.01735 76561 | 52 52 | 0.21321 17818 | 1.16009 27802 | 0.70039 72833 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.069042299609032$, $\Theta 0 = 0.8619608462$, $HK = 1.0300875730$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 - r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.32039 64540 | 0.00000 00000 | 90° 0′ | 2.03471 53122 | 90 |
| 0.99984 19155 | 1.32029 87371 | 0.00654 66917 | 89 15 | 2.01210 73643 | 89 |
| 0.99936 77261 | 1.32000 57060 | 0.01308 82806 | 88 31 | 1.98949 94164 | 88 |
| 0.99857 76238 | 1.31951 77192 | 0.01961 96606 | 87 46 | 1.96689 14685 | 87 |
| 0.99747 19280 | 1.31883 53734 | 0.02613 57182 | 87 1 | 1.94428 35205 | 86 |
| 0.99605 10861 | 1.31795 95033 | 0.03263 13295 | 86 17 | 1.92167 55726 | 85 |
| 0.99431 56720 | 1.31689 11801 | 0.03910 13564 | 85 32 | 1.89906 76247 | 84 |
| 0.99226 63864 | 1.31563 17106 | 0.04554 06434 | 84 47 | 1.87645 96768 | 83 |
| 0.98990 40553 | 1.31418 26349 | 0.05194 40144 | 84 2 | 1.85385 17289 | 82 |
| 0.98722 96302 | 1.31254 57253 | 0.05830 62693 | 83 17 | 1.83124 37810 | 81 |
| 0.98424 41861 | 1.31072 29838 | 0.06462 21812 | 82 32 | 1.80863 58331 | 80 |
| 0.98094 89213 | 1.30871 66392 | 0.07088 64934 | 81 46 | 1.78602 78851 | 79 |
| 0.97734 51558 | 1.30652 91449 | 0.07709 39167 | 81 1 | 1.76341 99372 | 78 |
| 0.97343 43300 | 1.30416 31759 | 0.08323 91270 | 80 15 | 1.74081 19893 | 77 |
| 0.96921 80039 | 1.30162 16250 | 0.08931 67629 | 79 29 | 1.71820 40414 | 76 |
| 0.96469 78546 | 1.29890 75994 | 0.09532 14240 | 78 43 | 1.69559 60935 | 75 |
| 0.95987 56758 | 1.29602 44173 | 0.10124 76688 | 77 56 | 1.67298 81456 | 74 |
| 0.95475 33753 | 1.29297 56032 | 0.10709 00133 | 77 10 | 1.65038 01977 | 73 |
| 0.94933 29736 | 1.28976 48840 | 0.11284 29301 | 76 23 | 1.62777 22497 | 72 |
| 0.94361 66021 | 1.28639 61840 | 0.11850 08473 | 75 35 | 1.60516 43018 | 71 |
| 0.93760 65006 | 1.28287 36204 | 0.12405 81487 | 74 48 | 1.58255 63539 | 70 |
| 0.93130 50161 | 1.27920 14980 | 0.12950 91731 | 74 0 | 1.55994 84060 | 69 |
| 0.92471 45998 | 1.27538 43041 | 0.13484 82153 | 73 12 | 1.53734 04581 | 68 |
| 0.91783 78055 | 1.27142 67027 | 0.14006 95267 | 72 23 | 1.51473 25102 | 67 |
| 0.91067 72870 | 1.26733 35291 | 0.14516 73172 | 71 35 | 1.49212 45623 | 66 |
| 0.90323 57961 | 1.26310 97835 | 0.15013 57566 | 70 46 | 1.46951 66144 | 65 |
| 0.89551 61797 | 1.25876 06253 | 0.15496 89777 | 69 56 | 1.44690 86665 | 64 |
| 0.88752 13778 | 1.25429 13663 | 0.15966 10790 | 69 7 | 1.42430 07185 | 63 |
| 0.87925 44206 | 1.24970 74646 | 0.16420 61290 | 68 16 | 1.40169 27706 | 62 |
| 0.87071 84265 | 1.24501 45176 | 0.16859 81701 | 67 26 | 1.37908 48227 | 61 |
| 0.86191 65988 | 1.24021 82552 | 0.17283 12244 | 66 35 | 1.35647 68748 | 60 |
| 0.85285 22237 | 1.23532 45329 | 0.17689 92991 | 65 43 | 1.33386 89269 | 59 |
| 0.84352 86672 | 1.23033 93242 | 0.18079 63935 | 64 51 | 1.31126 09790 | 58 |
| 0.83394 93726 | 1.22526 87137 | 0.18451 65064 | 63 59 | 1.28865 30311 | 57 |
| 0.82411 78578 | 1.22011 88895 | 0.18805 36444 | 63 6 | 1.26604 50832 | 56 |
| 0.81403 77126 | 1.21489 61356 | 0.19140 18312 | 62 12 | 1.24343 71353 | 55 |
| 0.80371 25960 | 1.20960 68240 | 0.19455 51177 | 61 19 | 1.22082 91873 | 54 |
| 0.79314 62334 | 1.20425 74072 | 0.19750 75927 | 60 24 | 1.19822 12394 | 53 |
| 0.78234 24136 | 1.19885 44102 | 0.20025 33955 | 59 30 | 1.17561 32915 | 52 |
| 0.77130 49868 | 1.19340 44225 | 0.20278 67279 | 58 35 | 1.15300 53436 | 51 |
| 0.76003 78612 | 1.18791 40899 | 0.20510 18688 | 57 39 | 1.13039 73957 | 50 |
| 0.74854 50007 | 1.18239 01066 | 0.20719 31885 | 56 42 | 1.10778 94478 | 49 |
| 0.73683 04220 | 1.17683 92068 | 0.20905 51650 | 55 46 | 1.08518 14999 | 48 |
| 0.72489 81922 | 1.17126 81567 | 0.21068 24001 | 54 48 | 1.06257 35519 | 47 |
| 0.71275 24260 | 1.16568 37461 | 0.21206 96376 | 53 50 | 1.03996 56041 | 46 |
| 0.70039 72833 | 1.16009 27802 | 0.21321 17818 | 52 52 | 1.01735 76561 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 2.1565156475, K′ = 1.6857503548, E = 1.211056028, E′ = 1.4674622093,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.02396 12850 | 1 22 | 0.01050 21636 | 1.00012 58452 | 0.01694 24822 |
| 2 | 0.04792 25699 | 2 45 | 0.02098 36904 | 1.00050 32288 | 0.03388 07351 |
| 3 | 0.07188 38549 | 4 7 | 0.03142 40274 | 1.00113 16945 | 0.05081 05279 |
| 4 | 0.09584 51399 | 5 29 | 0.04180 27880 | 1.00201 04822 | 0.06772 76275 |
| 5 | 0.11980 64248 | 6 51 | 0.05209 98337 | 1.00313 85295 | 0.08462 77970 |
| 6 | 0.14376 77098 | 8 13 | 0.06229 53533 | 1.00451 44723 | 0.10150 67944 |
| 7 | 0.16772 89948 | 9 35 | 0.07236 99392 | 1.00613 66468 | 0.11836 03717 |
| 8 | 0.19169 02798 | 10 56 | 0.08230 46606 | 1.00800 30911 | 0.13518 42734 |
| 9 | 0.21565 15647 | 12 17 | 0.09208 11326 | 1.01011 15480 | 0.15197 42358 |
| 10 | 0.23961 28497 | 13 38 | 0.10168 15801 | 1.01245 94672 | 0.16872 59855 |
| 11 | 0.26357 41347 | 14 58 | 0.11108 88976 | 1.01504 40088 | 0.18543 52386 |
| 12 | 0.28753 54197 | 16 18 | 0.12028 67034 | 1.01786 20463 | 0.20209 76999 |
| 13 | 0.31149 67046 | 17 38 | 0.12925 93879 | 1.02091 01701 | 0.21870 90619 |
| 14 | 0.33545 79896 | 18 57 | 0.13799 21563 | 1.02418 46923 | 0.23526 50037 |
| 15 | 0.35941 92746 | 20 16 | 0.14647 10652 | 1.02768 16504 | 0.25176 11911 |
| 16 | 0.38338 05595 | 21 35 | 0.15468 30530 | 1.03139 68120 | 0.26819 32750 |
| 17 | 0.40734 18445 | 22 53 | 0.16261 59647 | 1.03532 56803 | 0.28455 68916 |
| 18 | 0.43130 31295 | 24 10 | 0.17025 85702 | 1.03946 34991 | 0.30084 76617 |
| 19 | 0.45526 44145 | 25 26 | 0.17760 05773 | 1.04380 52583 | 0.31706 11903 |
| 20 | 0.47922 56994 | 26 42 | 0.18463 26382 | 1.04834 57003 | 0.33319 30665 |
| 21 | 0.50318 69844 | 27 58 | 0.19134 63517 | 1.05307 93260 | 0.34923 88634 |
| 22 | 0.52714 82694 | 29 13 | 0.19773 42593 | 1.05800 04010 | 0.36519 41381 |
| 23 | 0.55110 95544 | 30 27 | 0.20378 98371 | 1.06310 29632 | 0.38105 44318 |
| 24 | 0.57507 08393 | 31 41 | 0.20950 74827 | 1.06838 08291 | 0.39681 52701 |
| 25 | 0.59903 21243 | 32 54 | 0.21488 24988 | 1.07382 76019 | 0.41247 21633 |
| 26 | 0.62299 34093 | 34 7 | 0.21991 10718 | 1.07943 66784 | 0.42802 06069 |
| 27 | 0.64695 46942 | 35 18 | 0.22459 02484 | 1.08520 12575 | 0.44345 60826 |
| 28 | 0.67091 59792 | 36 29 | 0.22891 79082 | 1.09111 43480 | 0.45877 40585 |
| 29 | 0.69487 72642 | 37 39 | 0.23289 27342 | 1.09716 87771 | 0.47396 99905 |
| 30 | 0.71883 85492 | 38 49 | 0.23651 41807 | 1.10335 71989 | 0.48903 93230 |
| 31 | 0.74279 98341 | 39 58 | 0.23978 24399 | 1.10967 21031 | 0.50397 74905 |
| 32 | 0.76676 11191 | 41 6 | 0.24269 84060 | 1.11610 58243 | 0.51877 99184 |
| 33 | 0.79072 24041 | 42 13 | 0.24526 36394 | 1.12265 05510 | 0.53344 20249 |
| 34 | 0.81468 36890 | 43 20 | 0.24748 03283 | 1.12929 83350 | 0.54795 92224 |
| 35 | 0.83864 49740 | 44 26 | 0.24935 12513 | 1.13604 11010 | 0.56232 69191 |
| 36 | 0.86260 62590 | 45 31 | 0.25087 97387 | 1.14287 06563 | 0.57654 05212 |
| 37 | 0.88656 75440 | 46 35 | 0.25206 96336 | 1.14977 87007 | 0.59059 54347 |
| 38 | 0.91052 88289 | 47 39 | 0.25292 52540 | 1.15675 68364 | 0.60448 70673 |
| 39 | 0.93449 01139 | 48 42 | 0.25345 13545 | 1.16379 65783 | 0.61821 08313 |
| 40 | 0.95845 13989 | 49 44 | 0.25365 30884 | 1.17088 93642 | 0.63176 21451 |
| 41 | 0.98241 26838 | 50 45 | 0.25353 59713 | 1.17802 65652 | 0.64513 64364 |
| 42 | 1.00637 39688 | 51 46 | 0.25310 58450 | 1.18519 94959 | 0.65832 91446 |
| 43 | 1.03033 52538 | 52 46 | 0.25236 88429 | 1.19239 94253 | 0.67133 57232 |
| 44 | 1.05429 65388 | 53 45 | 0.25133 13558 | 1.19961 75873 | 0.68415 16433 |
| 45 | 1.07825 78237 | 54 44 | 0.25000 00000 | 1.20684 51910 | 0.69677 23959 |
| 90−r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 60°$

$q = 0.085795733702195$, $\Theta 0 = 0.8285168980$, $HK = 1.0903895588$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.41421 35624 | 0.00000 00000 | 90° 0′ | 2.15651 56475 | 90 |
| 0.99983 87925 | 1.41408 70799 | 0.00746 45017 | 89 19 | 2.13255 43625 | 89 |
| 0.99935 52434 | 1.41370 77878 | 0.01492 38646 | 88 38 | 2.10859 30775 | 88 |
| 0.99854 95732 | 1.41307 61515 | 0.02237 29430 | 87 57 | 2.08463 17926 | 87 |
| 0.99742 21491 | 1.41219 29466 | 0.02980 65777 | 87 16 | 2.06067 05076 | 86 |
| 0.99597 34843 | 1.41105 92570 | 0.03721 95889 | 86 35 | 2.03670 92226 | 85 |
| 0.99420 42378 | 1.40967 64744 | 0.04460 67701 | 85 53 | 2.01274 79377 | 84 |
| 0.99211 52135 | 1.40804 62958 | 0.05196 28815 | 85 11 | 1.98878 66527 | 83 |
| 0.98970 73588 | 1.40617 07222 | 0.05928 26440 | 84 29 | 1.96482 53677 | 82 |
| 0.98698 17641 | 1.40405 20551 | 0.06656 07336 | 83 47 | 1.94086 40827 | 81 |
| 0.98393 96610 | 1.40169 28947 | 0.07379 17757 | 83 5 | 1.91690 27978 | 80 |
| 0.98058 24210 | 1.39909 61356 | 0.08097 03401 | 82 23 | 1.89294 15128 | 79 |
| 0.97691 15541 | 1.39626 49639 | 0.08809 09364 | 81 41 | 1.86898 02278 | 78 |
| 0.97292 87065 | 1.39320 28531 | 0.09514 80095 | 80 58 | 1.84501 89429 | 77 |
| 0.96863 56591 | 1.38991 35592 | 0.10213 59353 | 80 15 | 1.82105 76579 | 76 |
| 0.96403 43250 | 1.38640 11169 | 0.10904 90175 | 79 32 | 1.79709 63729 | 75 |
| 0.95912 67478 | 1.38266 98339 | 0.11588 14840 | 78 49 | 1.77313 50879 | 74 |
| 0.95391 50985 | 1.37872 42853 | 0.12262 74837 | 78 5 | 1.74917 38030 | 73 |
| 0.94840 16738 | 1.37456 93090 | 0.12928 10844 | 77 21 | 1.72521 25180 | 72 |
| 0.94258 88926 | 1.37020 99983 | 0.13583 62697 | 76 37 | 1.70125 12330 | 71 |
| 0.93647 92941 | 1.36565 16965 | 0.14228 69378 | 75 53 | 1.67728 99480 | 70 |
| 0.93007 55342 | 1.36089 99899 | 0.14862 68991 | 75 8 | 1.65332 86631 | 69 |
| 0.92338 03829 | 1.35596 07006 | 0.15484 98749 | 74 23 | 1.62936 73781 | 68 |
| 0.91639 67210 | 1.35083 98797 | 0.16094 94967 | 73 37 | 1.60540 60931 | 67 |
| 0.90912 75372 | 1.34554 37995 | 0.16691 93054 | 72 51 | 1.58144 48082 | 66 |
| 0.90157 59245 | 1.34007 89457 | 0.17275 27505 | 72 5 | 1.55748 35232 | 65 |
| 0.89374 50771 | 1.33445 20094 | 0.17844 31913 | 71 18 | 1.53352 22382 | 64 |
| 0.88563 82868 | 1.32866 98789 | 0.18398 38964 | 70 30 | 1.50956 09532 | 63 |
| 0.87725 89396 | 1.32273 96308 | 0.18936 80462 | 69 42 | 1.48559 96683 | 62 |
| 0.86861 05122 | 1.31666 85215 | 0.19458 87340 | 68 54 | 1.46163 83833 | 61 |
| 0.85969 65682 | 1.31046 39783 | 0.19963 89691 | 68 5 | 1.43767 70983 | 60 |
| 0.85052 07549 | 1.30413 35898 | 0.20451 16802 | 67 16 | 1.41371 58134 | 59 |
| 0.84108 67990 | 1.29768 50969 | 0.20919 97204 | 66 26 | 1.38975 45284 | 58 |
| 0.83139 85036 | 1.29112 63832 | 0.21369 58722 | 65 36 | 1.36579 32434 | 57 |
| 0.82145 97438 | 1.28446 54650 | 0.21799 28546 | 64 45 | 1.34183 19584 | 56 |
| 0.81127 44636 | 1.27771 04815 | 0.22208 33313 | 63 53 | 1.31787 06735 | 55 |
| 0.80084 66719 | 1.27086 96850 | 0.22595 99196 | 63 1 | 1.29390 93885 | 54 |
| 0.79018 04386 | 1.26395 14305 | 0.22961 52018 | 62 9 | 1.26994 81035 | 53 |
| 0.77927 98915 | 1.25696 41655 | 0.23304 17372 | 61 15 | 1.24598 68185 | 52 |
| 0.76814 92120 | 1.24991 64194 | 0.23623 20761 | 60 21 | 1.22202 55336 | 51 |
| 0.75679 26317 | 1.24281 67937 | 0.23917 87758 | 59 27 | 1.19806 42486 | 50 |
| 0.74521 44290 | 1.23567 39504 | 0.24187 44177 | 58 32 | 1.17410 29636 | 49 |
| 0.73341 89253 | 1.22849 66025 | 0.24431 16265 | 57 36 | 1.15014 16787 | 48 |
| 0.72141 04816 | 1.22129 35025 | 0.24648 30908 | 56 39 | 1.12618 03937 | 47 |
| 0.70919 34952 | 1.21407 34320 | 0.24838 15864 | 55 42 | 1.10221 91087 | 46 |
| 0.69677 23959 | 1.20684 51910 | 0.25000 00000 | 54 44 | 1.07825 78237 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 2.3087867982, K′ = 1.6489952185, E = 1.1638279645, E′ = 1.4981149284,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.02565 31866 | 1 28 | 0.01271 71437 | 1.00016 31607 | 0.01667 62945 |
| 2 | 0.05130 63733 | 2 56 | 0.02540 65870 | 1.00065 24464 | 0.03334 89266 |
| 3 | 0.07695 95599 | 4 24 | 0.03804 07622 | 1.00146 72698 | 0.05001 42309 |
| 4 | 0.10261 27466 | 5 52 | 0.05059 23651 | 1.00260 66524 | 0.06666 85367 |
| 5 | 0.12826 59332 | 7 20 | 0.06303 44839 | 1.00406 92257 | 0.08330 81651 |
| 6 | 0.15391 91199 | 8 47 | 0.07534 07235 | 1.00585 32333 | 0.09992 94260 |
| 7 | 0.17957 23085 | 10 14 | 0.08748 53252 | 1.00795 65320 | 0.11652 86159 |
| 8 | 0.20522 54932 | 11 41 | 0.09944 32800 | 1.01037 65954 | 0.13310 20150 |
| 9 | 0.23087 86798 | 13 8 | 0.11119 04341 | 1.01311 05159 | 0.14964 58850 |
| 10 | 0.25653 18665 | 14 34 | 0.12270 35875 | 1.01615 50083 | 0.16615 64662 |
| 11 | 0.28218 50531 | 16 0 | 0.13396 05824 | 1.01950 64139 | 0.18262 99754 |
| 12 | 0.30783 82398 | 17 25 | 0.14494 03827 | 1.02316 07042 | 0.19906 26038 |
| 13 | 0.33349 14264 | 18 50 | 0.15562 31436 | 1.02711 34860 | 0.21545 05144 |
| 14 | 0.35914 46131 | 20 14 | 0.16599 02705 | 1.03136 00060 | 0.23178 98405 |
| 15 | 0.38479 77997 | 21 38 | 0.17602 44678 | 1.03589 51569 | 0.24807 66833 |
| 16 | 0.41045 09864 | 23 1 | 0.18570 97766 | 1.04071 34825 | 0.26430 71105 |
| 17 | 0.43610 41730 | 24 23 | 0.19503 16024 | 1.04580 91848 | 0.28047 71545 |
| 18 | 0.46175 73596 | 25 44 | 0.20397 67323 | 1.05117 61304 | 0.29658 28110 |
| 19 | 0.48741 05463 | 27 4 | 0.21253 33427 | 1.05680 78572 | 0.31260 00376 |
| 20 | 0 51306 37329 | 28 24 | 0.22069 09968 | 1.06269 75825 | 0.32858 47528 |
| 21 | 0.53871 69196 | 29 43 | 0.22844 06338 | 1.06883 82109 | 0.34447 28350 |
| 22 | 0.56437 01062 | 31 1 | 0.23577 45496 | 1.07522 23418 | 0.36028 01217 |
| 23 | 0.59002 32929 | 32 19 | 0.24268 63696 | 1.08184 22789 | 0.37600 24088 |
| 24 | 0.61567 64795 | 33 36 | 0.24917 10151 | 1.08869 00386 | 0.39163 54503 |
| 25 | 0.64132 96662 | 34 52 | 0.25522 46626 | 1.09575 73598 | 0.40717 49584 |
| 26 | 0.66698 28528 | 36 7 | 0.26084 46988 | 1.10303 57129 | 0.42261 66028 |
| 27 | 0.69263 60395 | 37 21 | 0.26602 96698 | 1.11051 63106 | 0.43795 60117 |
| 28 | 0.71828 92261 | 38 34 | 0.27077 92271 | 1.11819 01175 | 0.45318 87717 |
| 29 | 0.74394 24127 | 39 46 | 0.27509 40704 | 1.12604 78613 | 0.46831 04285 |
| 30 | 0.76959 55994 | 40 58 | 0.27897 58872 | 1.13408 00433 | 0.48331 64880 |
| 31 | 0.79524 87860 | 42 9 | 0.28242 72920 | 1.14227 69496 | 0.49820 24170 |
| 32 | 0.82090 19727 | 43 18 | 0.28545 17629 | 1.15062 86634 | 0.51296 36449 |
| 33 | 0.84655 51593 | 44 26 | 0.28805 35786 | 1.15912 50752 | 0.52759 55647 |
| 34 | 0.87220 83460 | 45 34 | 0.29023 77551 | 1.16775 58964 | 0.54209 35352 |
| 35 | 0.89786 15326 | 46 41 | 0.29200 99830 | 1.17651 06705 | 0.55645 28823 |
| 36 | 0.92351 47193 | 47 47 | 0.29337 65659 | 1.18537 87860 | 0.57066 89018 |
| 37 | 0.94916 79059 | 48 52 | 0.29434 43597 | 1.19434 94887 | 0.58473 68614 |
| 38 | 0.97482 10926 | 49 56 | 0.29492 07141 | 1.20341 18931 | 0.59865 20033 |
| 39 | 1.00047 42792 | 50 59 | 0.29511 34159 | 1.21255 50050 | 0.61240 95465 |
| 40 | 1.02612 74659 | 52 1 | 0.29493 06347 | 1.22176 77148 | 0.62600 46907 |
| 41 | 1.05178 06525 | 53 2 | 0.29438 08705 | 1.23103 88308 | 0.63943 26185 |
| 42 | 1.07743 38392 | 54 2 | 0.29347 29047 | 1.24035 70830 | 0.65268 84992 |
| 43 | 1.10308 70258 | 55 1 | 0.29221 57532 | 1.24971 11383 | 0.66576 74922 |
| 44 | 1.12874 02125 | 56 0 | 0.29061 86227 | 1.25908 96145 | 0.67866 47507 |
| 45 | 1.15439 33991 | 56 58 | 0.28869 08691 | 1.26848 10938 | 0.69137 54254 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.106054020185994$, $\Theta 0 = 0.7881449667$, $HK = 1.1541701350$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.53824 62687 | 0.00000 00000 | 90° 0′ | 2.30878 67982 | 90 |
| 0.99983 41412 | 1.53808 15440 | 0.00834 87781 | 88 23 | 2.28313 36115 | 89 |
| 0.99933 66526 | 1.53758 75740 | 0.01669 26008 | 88 46 | 2.25748 04249 | 88 |
| 0.99850 77970 | 1.53676 49688 | 0.02502 65041 | 88 9 | 2.23182 72382 | 87 |
| 0.99734 80125 | 1.53561 47447 | 0.03334 55075 | 87 32 | 2.20617 40516 | 86 |
| 0.99585 79109 | 1.53413 83232 | 0.04164 46052 | 86 54 | 2.18052 08649 | 85 |
| 0.99403 82778 | 1.53233 75281 | 0.04991 87582 | 86 16 | 2.15486 76783 | 84 |
| 0.99189 00707 | 1.53021 45843 | 0.05816 28855 | 85 38 | 2.12921 44916 | 83 |
| 0.98941 44182 | 1.52777 21140 | 0.06637 18564 | 85 0 | 2.10356 13050 | 82 |
| 0.98661 26176 | 1.52501 31340 | 0.07454 04819 | 84 22 | 2.07790 81184 | 81 |
| 0.98348 61339 | 1.52194 10514 | 0.08266 35068 | 83 44 | 2.05225 49317 | 80 |
| 0.98003 65970 | 1.51855 96596 | 0.09073 56016 | 83 6 | 2.02660 17451 | 79 |
| 0.97626 57996 | 1.51487 31329 | 0.09875 13547 | 82 27 | 2.00094 85584 | 78 |
| 0.97217 56947 | 1.51088 60218 | 0.10670 52642 | 81 48 | 1.97529 53718 | 77 |
| 0.96776 83924 | 1.50660 32466 | 0.11459 17308 | 81 9 | 1.94964 21851 | 76 |
| 0.96304 61576 | 1.50203 00916 | 0.12240 50500 | 80 30 | 1.92398 89985 | 75 |
| 0.95801 14060 | 1.49717 21977 | 0.13013 94047 | 79 50 | 1.89833 58118 | 74 |
| 0.95266 67013 | 1.49203 55559 | 0.13778 88583 | 79 10 | 1.87268 26251 | 73 |
| 0.94701 47511 | 1.48662 64993 | 0.14534 73477 | 78 30 | 1.84702 94385 | 72 |
| 0.94105 84035 | 1.48095 16947 | 0.15280 86769 | 77 49 | 1.82137 62519 | 71 |
| 0.93480 06429 | 1.47501 81348 | 0.16016 65105 | 77 8 | 1.79572 30652 | 70 |
| 0.92824 45859 | 1.46883 31288 | 0.16741 43683 | 76 26 | 1.77006 98786 | 69 |
| 0.92139 34772 | 1.46240 42933 | 0.17454 56190 | 75 44 | 1.74441 66919 | 68 |
| 0.91425 06851 | 1.45573 95424 | 0.18155 34763 | 75 2 | 1.71876 35053 | 67 |
| 0.90681 96968 | 1.44884 70781 | 0.18843 09933 | 74 19 | 1.69311 03186 | 66 |
| 0.89910 41140 | 1.44173 53793 | 0.19517 10594 | 73 36 | 1.66745 71320 | 65 |
| 0.89110 76479 | 1.43441 31916 | 0.20176 63966 | 72 52 | 1.64180 39453 | 64 |
| 0.88283 41144 | 1.42688 95162 | 0.20820 95570 | 72 8 | 1.61615 07587 | 63 |
| 0.87428 74294 | 1.41917 35981 | 0.21449 29211 | 71 23 | 1.59049 75721 | 62 |
| 0.86547 16034 | 1.41127 49149 | 0.22060 86968 | 70 37 | 1.56484 43854 | 61 |
| 0.85639 07366 | 1.40320 31647 | 0.22654 89197 | 69 51 | 1.53919 11988 | 60 |
| 0.84704 90138 | 1.39496 82541 | 0.23230 54536 | 69 4 | 1.51353 80121 | 59 |
| 0.83745 06991 | 1.38658 02852 | 0.23786 99932 | 68 17 | 1.48788 48255 | 58 |
| 0.82760 01310 | 1.37804 95440 | 0.24323 40676 | 67 29 | 1.46223 16388 | 57 |
| 0.81750 17168 | 1.36938 64865 | 0.24838 90447 | 66 41 | 1.43657 84522 | 56 |
| 0.80715 99276 | 1.36060 17261 | 0.25332 61379 | 65 52 | 1.41092 52655 | 55 |
| 0.79657 92934 | 1.35170 60205 | 0.25803 64133 | 65 2 | 1.38527 20789 | 54 |
| 0.78576 43973 | 1.34271 02582 | 0.26251 08001 | 64 11 | 1.35961 88922 | 53 |
| 0.77471 98708 | 1.33362 54449 | 0.26674 01012 | 63 20 | 1.33396 57055 | 52 |
| 0.76345 03889 | 1.32446 26900 | 0.27071 50065 | 62 28 | 1.30831 25189 | 51 |
| 0.75196 06646 | 1.31523 31927 | 0.27442 61086 | 61 35 | 1.28265 93322 | 50 |
| 0.74025 54443 | 1.30594 82284 | 0.27786 39198 | 60 41 | 1.25700 61456 | 49 |
| 0.72833 95027 | 1.29661 91348 | 0.28101 88920 | 59 46 | 1.23135 29589 | 48 |
| 0.71621 76383 | 1.28725 72976 | 0.28388 14388 | 58 51 | 1.20569 97723 | 47 |
| 0.70389 46686 | 1.27787 41372 | 0.28644 19600 | 57 55 | 1.18004 65856 | 46 |
| 0.69137 54254 | 1.26848 10938 | 0.28869 08691 | 56 58 | 1.15439 33991 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = 2.5045500790$, $K' = 1.6200258991$, $E = 1.1183777380$, $E' = 1.5237992053$,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.02782 83342 | 1 36 | 0.01539 55735 | 1.00021 42837 | 0.01627 42346 |
| 2 | 0.05565 66684 | 3 11 | 0.03075 31429 | 1.00085 68806 | 0.03254 56619 |
| 3 | 0.08348 50026 | 4 47 | 0.04603 49252 | 1.00192 70294 | 0.04881 14698 |
| 4 | 0.11131 33368 | 6 22 | 0.06120 35769 | 1.00342 34614 | 0.06506 88358 |
| 5 | 0.13914 16710 | 7 57 | 0.07622 24069 | 1.00534 44028 | 0.08131 49227 |
| 6 | 0.16697 00053 | 9 32 | 0.09105 55815 | 1.00768 75763 | 0.09754 68734 |
| 7 | 0.19479 83395 | 11 6 | 0.10566 83193 | 1.01045 02032 | 0.11376 18057 |
| 8 | 0.22262 66737 | 12 40 | 0.12002 70732 | 1.01362 90072 | 0.12995 68083 |
| 9 | 0.25045 50079 | 14 13 | 0.13409 96984 | 1.01722 02172 | 0.14612 89355 |
| 10 | 0.27828 33421 | 15 46 | 0.14785 56040 | 1.02121 95717 | 0.16227 52029 |
| 11 | 0.30611 16763 | 17 18 | 0.16126 58874 | 1.02562 23237 | 0.17839 25828 |
| 12 | 0.33394 00105 | 18 50 | 0.17430 34501 | 1.03042 32454 | 0.19447 80006 |
| 13 | 0.36176 83447 | 20 20 | 0.18694 30948 | 1.03561 66341 | 0.21052 83297 |
| 14 | 0.38959 66790 | 21 50 | 0.19916 16028 | 1.04119 63185 | 0.22654 03885 |
| 15 | 0.41742 50132 | 23 20 | 0.21093 77918 | 1.04715 56657 | 0.24251 09363 |
| 16 | 0.44525 33474 | 24 48 | 0 22225 25549 | 1.05348 75877 | 0.25843 66697 |
| 17 | 0.47308 16816 | 26 16 | 0.23308 88806 | 1.06018 45500 | 0.27431 42196 |
| 18 | 0.50091 00158 | 27 42 | 0.24343 18557 | 1.06723 85795 | 0.29014 01480 |
| 19 | 0.52873 83500 | 29 8 | 0.25326 86498 | 1.07464 12734 | 0.30591 09453 |
| 20 | 0.55656 66842 | 30 32 | 0.26258 84862 | 1.08238 38086 | 0.32162 30277 |
| 21 | 0.58439 50184 | 31 56 | 0.27138 25968 | 1.09045 69513 | 0.33727 27349 |
| 22 | 0.61222 33526 | 33 18 | 0.27964 41653 | 1.09885 10673 | 0.35285 63285 |
| 23 | 0.64005 16869 | 34 40 | 0.28736 82581 | 1.10755 61330 | 0.36836 99898 |
| 24 | 0.66788 00211 | 36 0 | 0.29455 17462 | 1.11656 17464 | 0.38380 98186 |
| 25 | 0.69570 83553 | 37 19 | 0.30119 32185 | 1.12585 71388 | 0.39917 18323 |
| 26 | 0.72353 66895 | 38 37 | 0.30729 28884 | 1.13543 11869 | 0.41445 19649 |
| 27 | 0.75136 50237 | 39 54 | 0.31285 24953 | 1.14527 24256 | 0.42964 60668 |
| 28 | 0.77919 33579 | 41 10 | 0.31787 52022 | 1.15536 90607 | 0.44474 99043 |
| 29 | 0.80702 16921 | 42 24 | 0.32236 54911 | 1.16570 89825 | 0.45975 91601 |
| 30 | 0.83485 00263 | 43 38 | 0.32632 90569 | 1.17627 97795 | 0.47466 94339 |
| 31 | 0.86267 83605 | 44 50 | 0.32977 27014 | 1.18706 87529 | 0.48947 62428 |
| 32 | 0.89050 66948 | 46 1 | 0.33270 42283 | 1.19806 29307 | 0.50417 50229 |
| 33 | 0.91833 50290 | 47 11 | 0.33513 23398 | 1.20924 90830 | 0.51876 11309 |
| 34 | 0.94616 33632 | 48 20 | 0.33706 65364 | 1.22061 37375 | 0.53322 98456 |
| 35 | 0.97399 16974 | 49 27 | 0.33851 70194 | 1.23214 31946 | 0.54757 63701 |
| 36 | 1.00182 00316 | 50 34 | 0.33949 45975 | 1.24382 35438 | 0.56179 58348 |
| 37 | 1.02964 83658 | 51 39 | 0.34001 05978 | 1.25564 06798 | 0.57588 32996 |
| 38 | 1.05747 67000 | 52 43 | 0.34007 67814 | 1.26758 03194 | 0.58983 37576 |
| 39 | 1.08530 50342 | 53 46 | 0.33970 52640 | 1.27962 80178 | 0.60364 21381 |
| 40 | 1.11313 33684 | 54 48 | 0.33890 84414 | 1.29176 91861 | 0.61730 33109 |
| 41 | 1.14096 17027 | 55 49 | 0.33769 89203 | 1.30398 91085 | 0.63081 20897 |
| 42 | 1.16879 00369 | 56 48 | 0.33608 94543 | 1.31627 29599 | 0.64416 32373 |
| 43 | 1.19661 83711 | 57 47 | 0.33409 28851 | 1.32860 58237 | 0.65735 14695 |
| 44 | 1.22444 67053 | 58 44 | 0.33172 20892 | 1.34097 27096 | 0.67037 14605 |
| 45 | 1.25227 50395 | 59 41 | 0.32898 99283 | 1.35335 85717 | 0.68321 78479 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 70°$

$q = 0.131061824499858$, $\Theta 0 = 0.7384664407$, $HK = 1.2240462555$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.70991 35651 | 0.00000 00000 | 90° 0′ | 2.50455 00790 | 90 |
| 0.99982 71058 | 1.70969 53883 | 0.00917 03805 | 89 27 | 2.47672 17448 | 89 |
| 0.99930 85325 | 1.70904 11308 | 0.01833 63062 | 88 55 | 2.44889 34106 | 88 |
| 0.99844 46074 | 1.70795 16110 | 0.02749 33119 | 88 22 | 2.42106 50764 | 87 |
| 0.99723 58755 | 1.70642 81917 | 0.03663 69110 | 87 49 | 2.39323 67422 | 86 |
| 0.99568 30984 | 1.70447 27784 | 0.04576 25853 | 87 16 | 2.36540 84079 | 85 |
| 0.99378 72533 | 1.70208 78163 | 0.05486 57745 | 86 43 | 2.33758 00737 | 84 |
| 0.99154 95309 | 1.69927 62875 | 0.06394 18650 | 86 10 | 2.30975 17395 | 83 |
| 0.98897 13334 | 1.69604 17067 | 0.07298 61798 | 85 36 | 2.28192 34053 | 82 |
| 0.98605 42725 | 1.69238 81168 | 0.08199 39678 | 85 3 | 2.25409 50711 | 81 |
| 0.98280 01661 | 1.68832 00831 | 0.09096 03928 | 84 29 | 2.22626 67369 | 80 |
| 0.97921 10356 | 1.68384 26872 | 0.09988 05231 | 83 55 | 2.19843 84027 | 79 |
| 0.97528 91023 | 1.67896 15207 | 0.10874 93206 | 83 21 | 2.17061 00685 | 78 |
| 0.97103 67835 | 1.67368 26771 | 0.11756 16303 | 82 46 | 2.14278 17343 | 77 |
| 0.96645 66885 | 1.66801 27439 | 0.12631 21691 | 82 12 | 2.11495 34000 | 76 |
| 0.96155 16144 | 1.66195 87940 | 0.13499 55158 | 81 37 | 2.08712 50658 | 75 |
| 0.95632 45409 | 1.65552 83761 | 0.14360 60995 | 81 1 | 2.05929 67316 | 74 |
| 0.95077 86259 | 1.64872 95046 | 0.15213 81898 | 80 25 | 2.03146 83974 | 73 |
| 0.94491 71996 | 1.64157 06491 | 0.16058 58855 | 79 49 | 2.00364 00632 | 72 |
| 0.93874 37597 | 1.63406 07230 | 0.16894 31044 | 79 13 | 1.97581 17290 | 71 |
| 0.93226 19647 | 1.62620 90720 | 0.17720 35729 | 78 36 | 1.94798 33948 | 70 |
| 0.92547 56289 | 1.61802 54615 | 0.18536 08158 | 77 58 | 1.92015 50606 | 69 |
| 0.91838 87155 | 1.60952 00637 | 0.19340 81461 | 77 20 | 1.89232 67264 | 68 |
| 0.91100 53304 | 1.60070 34445 | 0.20133 86551 | 76 42 | 1.86449 83921 | 67 |
| 0.90332 97156 | 1.59158 65494 | 0.20914 52034 | 76 3 | 1.83667 00579 | 66 |
| 0.89536 62423 | 1.58218 06891 | 0.21682 04110 | 75 23 | 1.80884 17237 | 65 |
| 0.88711 94043 | 1.57249 75252 | 0.22435 66494 | 74 43 | 1.78101 33895 | 64 |
| 0.87859 38106 | 1.56254 90544 | 0.23174 60328 | 74 2 | 1.75318 50553 | 63 |
| 0.86979 41783 | 1.55234 75933 | 0.23898 04111 | 73 21 | 1.72535 67211 | 62 |
| 0.86072 53257 | 1.54190 57623 | 0.24605 13624 | 72 39 | 1.69752 83869 | 61 |
| 0.85139 21644 | 1.53123 64694 | 0.25295 01875 | 71 56 | 1.66970 00527 | 60 |
| 0.84179 96923 | 1.52035 28933 | 0.25966 79043 | 71 13 | 1.64187 17185 | 59 |
| 0.83195 29861 | 1.50926 84668 | 0.26619 52443 | 70 29 | 1.61404 33842 | 58 |
| 0.82185 71938 | 1.49799 68595 | 0.27252 26492 | 69 44 | 1.58621 50500 | 57 |
| 0.81151 75269 | 1.48655 19601 | 0.27864 02697 | 68 59 | 1.55838 67158 | 56 |
| 0.80093 92537 | 1.47494 78592 | 0.28453 79654 | 68 12 | 1.53055 83816 | 55 |
| 0.79012 76914 | 1.46319 88308 | 0.29020 53069 | 67 25 | 1.50273 00474 | 54 |
| 0.77908 81986 | 1.45131 93148 | 0.29563 15786 | 66 37 | 1.47490 17132 | 53 |
| 0.76782 61683 | 1.43932 38985 | 0.30080 57852 | 65 48 | 1.44707 33790 | 52 |
| 0.75634 70207 | 1.42722 72983 | 0.30571 66593 | 64 59 | 1.41924 50448 | 51 |
| 0.74465 61957 | 1.41504 43413 | 0.31035 26720 | 64 8 | 1.39141 67106 | 50 |
| 0.73275 91466 | 1.40278 99470 | 0.31470 20462 | 63 17 | 1.36358 83763 | 49 |
| 0.72066 13327 | 1.39047 91083 | 0.31875 27727 | 62 24 | 1.33576 00421 | 48 |
| 0.70836 82126 | 1.37812 68735 | 0.32249 26298 | 61 31 | 1.30793 17079 | 47 |
| 0.69588 52382 | 1.36574 83271 | 0.32590 92064 | 60 36 | 1.28010 33737 | 46 |
| 0.68321 78479 | 1.35335 85717 | 0.32898 99283 | 59 41 | 1.25227 50395 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = 2.7680631454 = K'\sqrt{3}$, $K' = 1.5981420021$, $E = 1.076405113$, $E' = 1.5441504969$,

| r | Fφ | φ | E(r) | D(r) | A(r |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.03075 62572 | 1 46 | 0.01878 71553 | 1.00028 90226 | 0.01564 67728 |
| 2 | 0.06151 25143 | 3 37 | 0.03752 01201 | 1.00115 57568 | 0.03129 20711 |
| 3 | 0.09226 87715 | 5 17 | 0.05614 50985 | 1.00259 92025 | 0.04693 44040 |
| 4 | 0.12302 50287 | 7 2 | 0.07460 90790 | 1.00461 76935 | 0.06257 22754 |
| 5 | 0.15378 12859 | 8 47 | 0.09286 02109 | 1.00720 88997 | 0.07820 41558 |
| 6 | 0.18453 75430 | 10 31 | 0.11084 81632 | 1.01036 98288 | 0.09382 84843 |
| 7 | 0.21529 38002 | 12 15 | 0.12852 44620 | 1.01409 68295 | 0.10944 36574 |
| 8 | 0.24605 00574 | 13 58 | 0.14584 27986 | 1.01838 55946 | 0.12504 80220 |
| 9 | 0.27680 63145 | 15 40 | 0.16275 93073 | 1.02323 11658 | 0.14063 98665 |
| 10 | 0.30756 25717 | 17 22 | 0.17923 28093 | 1.02862 79374 | 0.15621 74137 |
| 11 | 0.33831 88289 | 19 3 | 0.19522 50184 | 1.03456 96626 | 0.17177 88130 |
| 12 | 0.36907 50860 | 20 43 | 0.21070 07095 | 1.04104 94593 | 0.18732 21327 |
| 13 | 0.39983 13432 | 22 22 | 0.22562 78479 | 1.04805 98163 | 0.20284 53538 |
| 14 | 0.43058 76004 | 23 59 | 0.23997 76797 | 1.05559 26010 | 0.21834 63622 |
| 15 | 0.46134 38576 | 25 36 | 0.25372 47838 | 1.06363 90673 | 0.23382 29430 |
| 16 | 0.49210 01147 | 27 12 | 0.26684 70884 | 1.07218 98642 | 0.24927 27739 |
| 17 | 0.52285 63719 | 28 46 | 0.27932 58519 | 1.08123 50446 | 0.26469 34194 |
| 18 | 0.55361 26291 | 30 19 | 0.29114 56129 | 1.09076 40755 | 0.28008 23255 |
| 19 | 0.58436 88862 | 31 50 | 0.30229 41110 | 1.10076 58484 | 0.29543 68145 |
| 20 | 0.61512 51434 | 33 21 | 0.31276 21816 | 1.11122 86903 | 0.31075 40803 |
| 21 | 0.64588 14006 | 34 50 | 0.32254 36297 | 1.12214 03756 | 0.32603 11842 |
| 22 | 0.67663 76577 | 36 17 | 0.33163 50828 | 1.13348 81382 | 0.34126 50509 |
| 23 | 0.70739 39149 | 37 43 | 0.34003 58309 | 1.14525 86847 | 0.35645 24653 |
| 24 | 0.73815 01721 | 39 8 | 0.34774 76532 | 1.15743 82078 | 0.37159 00694 |
| 25 | 0.76890 64293 | 40 31 | 0.35477 46364 | 1.17001 24008 | 0.38667 43599 |
| 26 | 0.79966 26864 | 41 52 | 0.36112 29881 | 1.18296 64722 | 0.40170 16862 |
| 27 | 0.83041 89436 | 43 12 | 0.36680 08467 | 1.19628 51612 | 0.41666 82489 |
| 28 | 0.86117 52008 | 44 31 | 0.37181 80918 | 1.20995 27538 | 0.43157 00988 |
| 29 | 0.89193 14579 | 45 48 | 0.37618 61563 | 1.22395 30995 | 0.44640 31361 |
| 30 | 0.92268 77151 | 47 3 | 0.37991 78428 | 1.23826 96285 | 0.46116 31110 |
| 31 | 0.95344 39723 | 48 18 | 0.38302 71460 | 1.25288 53692 | 0.47584 56238 |
| 32 | 0.98420 02294 | 49 30 | 0.38552 90817 | 1.26778 29672 | 0.49044 61259 |
| 33 | 1.01495 64866 | 50 41 | 0.38743 95246 | 1.28294 47038 | 0.50495 99214 |
| 34 | 1.04571 27438 | 51 51 | 0.38877 50552 | 1.29835 25154 | 0.51938 21695 |
| 35 | 1.07646 90010 | 52 59 | 0.38955 28159 | 1.31398 80140 | 0.53370 78866 |
| 36 | 1.10722 52581 | 54 5 | 0.38979 03785 | 1.32983 25072 | 0.54793 19494 |
| 37 | 1.13798 15153 | 55 10 | 0.38950 56204 | 1.34586 70195 | 0.56204 90989 |
| 38 | 1.16873 77725 | 56 14 | 0.38871 66125 | 1.36207 23140 | 0.57605 39442 |
| 39 | 1.19949 40296 | 57 16 | 0.38744 15171 | 1.37842 89138 | 0.58994 09669 |
| 40 | 1.23025 02868 | 58 17 | 0.38569 84955 | 1.39491 71251 | 0.60370 45267 |
| 41 | 1.26100 65440 | 59 17 | 0.38350 56260 | 1.41151 70596 | 0.61733 88663 |
| 42 | 1.29176 28011 | 60 15 | 0.38088 08305 | 1.42820 86579 | 0.63083 81179 |
| 43 | 1.32251 90583 | 61 12 | 0.37784 18107 | 1.44497 17132 | 0.64419 63092 |
| 44 | 1.35327 53155 | 62 8 | 0.37440 59923 | 1.46178 58952 | 0.65740 73705 |
| 45 | 1.38403 15727 | 63 2 | 0.37059 04774 | 1.47863 07744 | 0.67046 51423 |
| 90−r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 75°$

$q = 0.163033534821580$, $\Theta 0 = 0.6753457533$, $HK = 1.3046678096$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 1.96563 05108 | 0.00000 00000 | 90° 0′ | 2.76806 31454 | 90 |
| 0.99981 60886 | 1.96533 12951 | 0.00989 91720 | 89 33 | 2.73730 68882 | 89 |
| 0.99926 44975 | 1.96443 40309 | 0.01979 47043 | 89 5 | 2.70655 06310 | 88 |
| 0.99834 56552 | 1.96293 98674 | 0.02968 29453 | 88 38 | 2.67579 43738 | 87 |
| 0.99706 02753 | 1.96085 07176 | 0.03956 02195 | 88 10 | 2.64503 81167 | 86 |
| 0.99540 93546 | 1.95816 92561 | 0.04942 28154 | 87 43 | 2.61428 18595 | 85 |
| 0.99339 41714 | 1.95489 89147 | 0.05926 69738 | 87 15 | 2.58352 56023 | 84 |
| 0.99101 62829 | 1.95104 38778 | 0.06908 88752 | 86 47 | 2.55276 93451 | 83 |
| 0.98827 75221 | 1.94660 90763 | 0.07888 46278 | 86 19 | 2.52201 30880 | 82 |
| 0.98517 99940 | 1.94160 01803 | 0.08865 02550 | 85 51 | 2.49125 68308 | 81 |
| 0.98172 60720 | 1.93602 35909 | 0.09838 16828 | 85 22 | 2.46050 05736 | 80 |
| 0.97791 83923 | 1.92988 64309 | 0.10807 47268 | 84 54 | 2.42974 43165 | 79 |
| 0.97375 98498 | 1.92319 65349 | 0.11772 50798 | 84 25 | 2.39898 80593 | 78 |
| 0.96925 35914 | 1.91596 24373 | 0.12732 82981 | 83 55 | 2.36823 18021 | 77 |
| 0.96440 30106 | 1.90819 33609 | 0.13687 97883 | 83 26 | 2.33747 55450 | 76 |
| 0.95921 17405 | 1.89989 92030 | 0.14637 47936 | 82 56 | 2.30671 92878 | 75 |
| 0.95368 36468 | 1.89109 05214 | 0.15580 83802 | 82 25 | 2.27596 30306 | 74 |
| 0.94782 28200 | 1.88177 85195 | 0.16517 54225 | 81 55 | 2.24520 67734 | 73 |
| 0.94163 35686 | 1.87197 50301 | 0.17447 05894 | 81 24 | 2.21445 05163 | 72 |
| 0.93512 04092 | 1.86169 24991 | 0.18368 83293 | 80 52 | 2.18369 42591 | 71 |
| 0.92828 80593 | 1.85094 39670 | 0.19282 28550 | 80 20 | 2.15293 80019 | 70 |
| 0.92114 14274 | 1.83974 30516 | 0.20186 81293 | 79 48 | 2.12218 17448 | 69 |
| 0.91368 56040 | 1.82810 39279 | 0.21081 78488 | 79 15 | 2.09142 54876 | 68 |
| 0.90592 58521 | 1.81604 13089 | 0.21966 54291 | 78 41 | 2.06066 92304 | 67 |
| 0.89786 75972 | 1.80357 04247 | 0.22840 39887 | 78 7 | 2.02991 29733 | 66 |
| 0.88951 64174 | 1.79070 70015 | 0.23702 63334 | 77 32 | 1.99915 67161 | 65 |
| 0.88087 80328 | 1.77746 72401 | 0.24552 49406 | 76 56 | 1.96840 04589 | 64 |
| 0.87195 82952 | 1.76386 77929 | 0.25389 19433 | 76 20 | 1.93764 42017 | 63 |
| 0.86276 31773 | 1.74992 57419 | 0.26211 91147 | 75 43 | 1.90688 79446 | 62 |
| 0.85329 87622 | 1.73565 85746 | 0.27019 78524 | 75 6 | 1.87613 16874 | 61 |
| 0.84357 12322 | 1.72108 41609 | 0.27811 91636 | 74 27 | 1.84537 54302 | 60 |
| 0.83358 68580 | 1.70622 07286 | 0.28587 36500 | 73 48 | 1.81461 91731 | 59 |
| 0.82335 19876 | 1.69108 68389 | 0.29345 14936 | 73 8 | 1.78386 29159 | 58 |
| 0.81287 30353 | 1.67570 13618 | 0.30084 24433 | 72 28 | 1.75310 66587 | 57 |
| 0.80215 64710 | 1.66008 34507 | 0.30803 58026 | 71 46 | 1.72235 04016 | 56 |
| 0.79120 88085 | 1.64425 25175 | 0.31502 04176 | 71 4 | 1.69159 41444 | 55 |
| 0.78003 65955 | 1.62822 82065 | 0.32178 46673 | 70 20 | 1.66083 78872 | 54 |
| 0.76864 64021 | 1.61203 03692 | 0.32831 64547 | 69 36 | 1.63008 16300 | 53 |
| 0.75704 48103 | 1.59567 90385 | 0.33460 32006 | 68 50 | 1.59932 53729 | 52 |
| 0.74523 84036 | 1.57919 44025 | 0.34063 18384 | 68 4 | 1.56856 91157 | 51 |
| 0.73323 37566 | 1.56259 67789 | 0.34638 88130 | 67 16 | 1.53781 28585 | 50 |
| 0.72103 74248 | 1.54590 65890 | 0.35186 00808 | 66 28 | 1.50705 66014 | 49 |
| 0.70865 59347 | 1.52914 43320 | 0.35703 11148 | 65 38 | 1.47630 03442 | 48 |
| 0.69609 57739 | 1.51233 05588 | 0.36188 69115 | 64 47 | 1.44554 40870 | 47 |
| 0.68336 33823 | 1.49548 58469 | 0.36641 20039 | 63 55 | 1.41478 78299 | 46 |
| 0.67046 51423 | 1.47863 07744 | 0.37059 04774 | 63 2 | 1.38403 15727 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = 3.1533852519$, $K' = 1.5828428043$, $E = 1.0401143957$, $E' = 1.5588871966$,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.03503 76139 | 2 0 | 0.02346 68886 | 1.00041 13182 | 0.01460 06854 |
| 2 | 0.07007 52278 | 4 1 | 0.04685 05457 | 1.00164 48264 | 0.02920 20956 |
| 3 | 0.10511 28417 | 6 1 | 0.07006 85417 | 1.00369 91860 | 0.04380 49412 |
| 4 | 0.14015 04556 | 8 0 | 0.09304 00333 | 1.00657 21668 | 0.05840 99043 |
| 5 | 0.17518 80695 | 9 59 | 0.11568 65173 | 1.01026 06485 | 0.07301 76251 |
| 6 | 0.21022 56835 | 11 58 | 0.13793 25365 | 1.01476 06225 | 0.08762 86871 |
| 7 | 0.24526 32974 | 13 55 | 0.15970 63263 | 1.02006 71948 | 0.10224 36040 |
| 8 | 0.28030 09113 | 15 52 | 0.18094 03901 | 1.02617 45886 | 0.11686 28061 |
| 9 | 0.31533 85252 | 17 47 | 0.20157 19949 | 1.03307 61484 | 0.13148 66263 |
| 10 | 0.35037 61391 | 19 41 | 0.22154 35813 | 1.04076 43440 | 0.14611 52882 |
| 11 | 0.38541 37530 | 21 34 | 0.24080 30831 | 1.04923 07759 | 0.16074 88922 |
| 12 | 0.42045 13669 | 23 26 | 0.25930 41559 | 1.05846 61800 | 0.17538 74040 |
| 13 | 0.45548 89808 | 25 16 | 0.27700 63163 | 1.06846 04345 | 0.19003 06422 |
| 14 | 0.49052 65947 | 27 4 | 0.29387 49943 | 1.07920 25667 | 0.20467 82669 |
| 15 | 0.52556 42086 | 28 51 | 0.30988 15035 | 1.09068 07598 | 0.21932 97686 |
| 16 | 0.56060 18226 | 30 36 | 0.32500 29380 | 1.10288 23622 | 0.23398 44577 |
| 17 | 0.59563 94365 | 32 20 | 0.33922 20017 | 1.11579 38955 | 0.24864 14540 |
| 18 | 0.63067 70504 | 34 1 | 0.35252 67798 | 1.12940 10647 | 0.26329 96779 |
| 19 | 0.66571 46643 | 35 41 | 0.36491 04618 | 1.14368 87684 | 0.27795 78408 |
| 20 | 0.70075 22782 | 37 18 | 0.37637 10249 | 1.15864 11101 | 0.29261 44375 |
| 21 | 0.73578 98921 | 38 54 | 0.38691 08879 | 1.17424 14105 | 0.30726 77376 |
| 22 | 0.77082 75060 | 40 28 | 0.39653 65430 | 1.19047 22196 | 0.32191 57797 |
| 23 | 0.80586 51199 | 41 59 | 0.40525 81757 | 1.20731 53312 | 0.33655 63638 |
| 24 | 0.84090 27338 | 43 29 | 0.41308 92784 | 1.22475 17970 | 0.35118 70467 |
| 25 | 0.87594 03477 | 44 56 | 0.42004 62655 | 1.24276 19421 | 0.36580 51367 |
| 26 | 0.91097 79617 | 46 22 | 0.42614 80965 | 1.26132 53814 | 0.38040 76896 |
| 27 | 0.94601 55756 | 47 45 | 0.43141 59095 | 1.28042 10369 | 0.39499 15050 |
| 28 | 0.98105 31895 | 49 7 | 0.43587 26721 | 1.30002 71557 | 0.40955 31244 |
| 29 | 1.01609 08034 | 50 26 | 0.43954 28505 | 1.32012 13294 | 0.42408 88287 |
| 30 | 1.05112 84173 | 51 44 | 0.44245 21005 | 1.34068 05139 | 0.43859 46375 |
| 31 | 1.08616 60312 | 52 59 | 0.44462 69813 | 1.36168 10508 | 0.45306 63090 |
| 32 | 1.12120 36451 | 54 12 | 0.44609 46931 | 1.38309 86893 | 0.46749 93405 |
| 33 | 1.15624 12590 | 55 24 | 0.44688 28394 | 1.40490 86089 | 0.48188 89699 |
| 34 | 1.19127 88729 | 56 33 | 0.44701 92128 | 1.42708 54443 | 0.49623 01775 |
| 35 | 1.22631 64868 | 57 41 | 0.44653 16053 | 1.44960 33094 | 0.51051 76900 |
| 36 | 1.26135 41008 | 58 47 | 0.44544 76404 | 1.47243 58241 | 0.52474 59832 |
| 37 | 1.29639 17147 | 59 51 | 0.44379 46284 | 1.49555 61410 | 0.53890 92878 |
| 38 | 1.33142 93286 | 60 53 | 0.44159 94403 | 1.51893 69731 | 0.55300 15938 |
| 39 | 1.36646 69425 | 61 54 | 0.43888 84024 | 1.54253 06233 | 0.56701 66575 |
| 40 | 1.40150 45564 | 62 53 | 0.43568 72080 | 1.56636 90138 | 0.58094 80084 |
| 41 | 1.43654 21703 | 63 50 | 0.43202 08450 | 1.59036 37173 | 0.59478 89567 |
| 42 | 1.47157 97842 | 64 45 | 0.42791 35381 | 1.61450 59885 | 0.60853 26019 |
| 43 | 1.50661 73981 | 65 39 | 0 42338 87053 | 1.63876 67967 | 0.62217 18423 |
| 44 | 1.54165 50120 | 66 32 | 0.41846 89243 | 1.66311 68595 | 0.63569 93846 |
| 45 | 1.57669 26259 | 67 23 | 0.41317 59112 | 1.68752 66770 | 0.64910 77548 |
| 90−r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.206609755200965$, $\Theta 0 = 0.590423578356$, $HK = 1.406061468420$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 2.39974 38370 | 0.00000 00000 | 90° 0′ | 3.15338 52519 | 90 |
| 0.99979 75549 | 2.39930 24464 | 0.01049 98939 | 89 39 | 3.11834 76380 | 89 |
| 0.99919 04200 | 2.39797 88675 | 0.02099 72691 | 89 18 | 3.08331 00241 | 88 |
| 0.99817 91961 | 2.39577 48778 | 0.03148 95952 | 88 57 | 3.04827 24102 | 87 |
| 0.99676 48832 | 2.39269 34364 | 0.04197 43187 | 88 36 | 3.01323 47963 | 86 |
| 0.99494 88778 | 2.38873 86793 | 0.05244 88508 | 88 15 | 2.97819 71823 | 85 |
| 0.99273 29703 | 2.38391 59122 | 0.06291 05559 | 87 54 | 2 94315 95684 | 84 |
| 0.99011 93406 | 2.37823 16019 | 0.07335 67394 | 87 32 | 2.90812 19545 | 83 |
| 0.98711 05534 | 2.37169 33654 | 0.08378 46353 | 87 11 | 2.87308 43406 | 82 |
| 0.98370 95524 | 2.36430 99572 | 0.09419 13935 | 86 49 | 2.83804 67267 | 81 |
| 0.97991 96536 | 2.35609 12550 | 0.10457 40674 | 86 27 | 2.80300 91128 | 80 |
| 0.97574 45380 | 2.34704 82431 | 0.11492 96001 | 86 4 | 2.76797 14989 | 79 |
| 0.97118 82434 | 2.33719 29943 | 0.12525 48110 | 85 42 | 2.73293 38850 | 78 |
| 0.96625 51552 | 2.32653 86504 | 0.13554 63814 | 85 19 | 2.69789 62711 | 77 |
| 0.96094 99971 | 2.31509 94002 | 0.14580 08404 | 84 56 | 2.66285 86572 | 76 |
| 0.95527 78200 | 2.30289 04563 | 0.15601 45490 | 84 32 | 2.62782 10432 | 75 |
| 0.94924 39913 | 2.28992 80308 | 0.16618 36848 | 84 8 | 2.59278 34293 | 74 |
| 0.94285 41832 | 2.27622 93087 | 0.17630 42256 | 83 44 | 2.55774 58154 | 73 |
| 0.93611 43595 | 2.26181 24201 | 0.18637 19320 | 83 19 | 2.52270 82015 | 72 |
| 0.92903 07633 | 2.24669 64112 | 0.19638 23298 | 82 54 | 2.48767 05876 | 71 |
| 0.92160 99031 | 2.23090 12139 | 0.20633 06915 | 82 28 | 2.45263 29137 | 70 |
| 0.91385 85385 | 2.21444 76139 | 0.21621 20167 | 82 1 | 2.41759 53578 | 69 |
| 0.90578 36660 | 2.19735 72184 | 0.22602 10124 | 81 35 | 2.38255 77459 | 68 |
| 0.89739 25035 | 2.17965 24214 | 0.23575 20713 | 81 7 | 2.34752 01320 | 67 |
| 0.88869 24749 | 2.16135 63692 | 0.24539 92508 | 80 39 | 2.31248 25181 | 66 |
| 0.87969 11946 | 2.14249 29245 | 0.25495 62494 | 80 10 | 2.27744 49041 | 65 |
| 0.87039 64511 | 2.12308 66296 | 0.26441 63838 | 79 41 | 2.24240 72902 | 64 |
| 0.86081 61906 | 2.10316 26690 | 0.27377 25638 | 79 11 | 2.20736 96763 | 63 |
| 0.85095 85006 | 2.08274 68307 | 0.28301 72673 | 78 40 | 2.17233 20624 | 62 |
| 0.84083 15928 | 2.06186 54682 | 0.29214 25142 | 78 8 | 2.13729 44485 | 61 |
| 0.83044 37863 | 2.04054 54606 | 0.30113 98388 | 77 35 | 2.10225 68346 | 60 |
| 0.81980 34906 | 2.01881 41730 | 0.31000 02630 | 77 2 | 2.06721 92207 | 59 |
| 0.80891 91886 | 1.99669 94165 | 0.31871 42670 | 76 28 | 2.03218 16068 | 58 |
| 0.79779 94194 | 1.97422 94075 | 0.32727 17611 | 75 52 | 1.99714 39929 | 57 |
| 0.78645 27612 | 1.95143 27275 | 0.33566 20561 | 75 16 | 1.96210 63790 | 56 |
| 0.77488 78149 | 1.92833 82823 | 0.34387 38337 | 74 39 | 1.92706 87650 | 55 |
| 0.76311 31867 | 1.90497 52611 | 0.35189 51171 | 74 1 | 1.89203 11511 | 54 |
| 0.75113 74717 | 1.88137 30959 | 0.35971 32414 | 73 21 | 1.85699 35372 | 53 |
| 0.73896 92379 | 1.85756 14210 | 0.36731 48250 | 72 41 | 1.82195 59233 | 52 |
| 0.72661 70097 | 1.83357 00328 | 0.37468 57413 | 71 59 | 1 78691 83094 | 51 |
| 0.71408 92524 | 1.80942 88493 | 0.38181 10919 | 71 16 | 1.75188 06955 | 50 |
| 0.70139 43563 | 1.78516 78703 | 0.38867 51812 | 70 32 | 1.71684 30816 | 49 |
| 0.68854 06225 | 1.76081 71386 | 0.39526 14938 | 69 47 | 1.68180 54677 | 48 |
| 0.67553 62475 | 1.73640 67003 | 0.40155 26735 | 69 0 | 1.64676 78538 | 47 |
| 0.66238 93095 | 1.71196 65668 | 0.40753 05071 | 68 12 | 1.61173 02399 | 46 |
| 0.64910 77548 | 1.68752 66770 | 0.41317 59112 | 67 23 | 1.57669 26259 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = 3.2553029421$, $K' = 1.5805409339$, $E = 1.033789462$, $E' = 1.5611417453$,

| r | F$\phi$ | $\phi$ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.03617 00327 | 2 4 | 0.02466 81037 | 1.00044 63617 | 0.01430 61216 |
| 2 | 0.07234 00654 | 4 8 | 0.04924 41210 | 1.00178 49728 | 0.02861 35824 |
| 3 | 0.10851 00981 | 6 12 | 0.07363 69132 | 1.00401 44114 | 0.04292 37056 |
| 4 | 0.14468 01308 | 8 16 | 0.09775 72158 | 1.00713 23089 | 0.05723 77835 |
| 5 | 0.18085 01635 | 10 18 | 0.12151 85252 | 1.01113 53504 | 0.07155 70609 |
| 6 | 0.21702 01961 | 12 20 | 0.14483 79258 | 1.01601 92772 | 0.08588 27206 |
| 7 | 0.25319 02288 | 14 21 | 0.16763 68426 | 1.02177 88885 | 0.10021 58677 |
| 8 | 0.28936 02615 | 16 21 | 0.18984 17049 | 1.02840 80440 | 0.11455 75144 |
| 9 | 0.32553 02942 | 18 20 | 0.21138 45101 | 1.03589 96677 | 0.12890 85656 |
| 10 | 0.36170 03269 | 20 18 | 0.23220 32821 | 1.04424 57511 | 0.14326 98042 |
| 11 | 0.39787 03596 | 22 14 | 0.25224 24183 | 1.05343 73577 | 0.15764 18767 |
| 12 | 0.43404 03923 | 24 8 | 0.27145 29257 | 1.06346 46282 | 0.17202 52803 |
| 13 | 0.47021 04250 | 26 1 | 0.28979 25485 | 1.07431 67854 | 0.18642 03484 |
| 14 | 0.50638 04577 | 27 53 | 0.30722 57913 | 1.08598 21410 | 0.20082 72392 |
| 15 | 0.54255 04904 | 29 42 | 0.32372 38467 | 1.09844 81017 | 0.21524 59210 |
| 16 | 0.57872 05230 | 31 29 | 0.33926 44357 | 1.11170 11775 | 0.22967 61638 |
| 17 | 0.61489 05557 | 33 15 | 0.35383 15704 | 1.12572 69891 | 0.24411 75248 |
| 18 | 0.65106 05884 | 34 58 | 0.36741 52534 | 1.14051 02773 | 0.25856 93397 |
| 19 | 0.68723 06211 | 36 40 | 0.38001 11223 | 1.15603 49127 | 0.27303 07120 |
| 20 | 0.72340 06538 | 38 19 | 0.39162 00536 | 1.17228 39058 | 0.28750 05037 |
| 21 | 0.75957 06865 | 39 56 | 0.40224 77358 | 1.18923 94189 | 0.30197 73269 |
| 22 | 0.79574 07192 | 41 32 | 0.41190 42239 | 1.20688 27779 | 0.31645 95358 |
| 23 | 0.83191 07519 | 43 4 | 0.42060 34838 | 1.22519 44855 | 0.33094 52195 |
| 24 | 0.86808 07846 | 44 35 | 0.42836 29362 | 1.24415 42355 | 0.34543 21958 |
| 25 | 0.90425 08173 | 46 4 | 0.43520 30077 | 1.26374 09274 | 0.35991 80053 |
| 26 | 0.94042 08500 | 47 30 | 0.44114 66947 | 1.28393 26825 | 0.37439 99070 |
| 27 | 0.97659 08826 | 48 54 | 0.44621 91466 | 1.30470 68611 | 0.38887 48743 |
| 28 | 1.01276 09153 | 50 16 | 0.45044 72717 | 1.32604 00803 | 0.40333 95918 |
| 29 | 1.04893 09480 | 51 36 | 0.45385 93683 | 1.34790 82334 | 0.41779 04532 |
| 30 | 1.08510 09807 | 52 54 | 0.45648 47848 | 1.37028 65097 | 0.43222 35599 |
| 31 | 1.12127 10134 | 54 9 | 0.45835 36084 | 1.39314 94160 | 0.44663 47209 |
| 32 | 1.15744 10461 | 55 23 | 0.45949 63831 | 1.41647 07992 | 0.46101 94525 |
| 33 | 1.19361 10788 | 56 34 | 0.45994 38581 | 1.44022 38696 | 0.47537 29805 |
| 34 | 1.22978 11115 | 57 43 | 0.45972 67648 | 1.46438 12257 | 0.48969 02419 |
| 35 | 1.26595 11442 | 58 51 | 0.45887 56209 | 1.48891 48802 | 0.50396 58883 |
| 36 | 1.30212 11769 | 59 56 | 0.45742 05619 | 1.51379 62870 | 0.51819 42896 |
| 37 | 1.33829 12095 | 61 0 | 0.45539 11968 | 1.53899 63693 | 0.53236 95393 |
| 38 | 1.37446 12422 | 62 2 | 0.45281 64872 | 1.56448 55491 | 0.54648 54602 |
| 39 | 1.41063 12749 | 63 1 | 0.44972 46468 | 1.59023 37776 | 0.56053 56107 |
| 40 | 1.44680 13076 | 64 0 | 0.44614 30615 | 1.61621 05676 | 0.57451 32929 |
| 41 | 1.48297 13403 | 64 56 | 0.44209 82256 | 1.64238 50248 | 0.58841 15607 |
| 42 | 1.51914 13730 | 65 51 | 0.43761 56944 | 1.66872 58833 | 0.60222 32286 |
| 43 | 1.55531 14057 | 66 44 | 0.43272 00503 | 1.69520 15399 | 0.61594 08825 |
| 44 | 1.59148 14384 | 67 35 | 0.42743 48807 | 1.72178 00903 | 0.62955 68896 |
| 45 | 1.62765 14711 | 68 25 | 0.42178 27675 | 1.74842 93662 | 0.64306 34108 |
| 90-r | F$\psi$ | $\psi$ | G(r) | C(r) | B(r) |

TABLE $\theta = 81°$

$q = 0.217548949699726$, $\Theta 0 = 0.5693797108$, $HK = 1.4306906219$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90-r |
|---|---|---|---|---|---|
| 1.00000 00000 | 2.52833 01251 | 0.00000 00000 | 90° 0′ | 3.25530 29421 | 90 |
| 0.99979 22836 | 2.52784 54320 | 0.01060 10292 | 89 41 | 3.21913 29095 | 89 |
| 0.99916 93515 | 2.52639 20136 | 0.02119 97963 | 89 21 | 3.18296 28768 | 88 |
| 0.99813 18540 | 2.52397 18509 | 0.03179 40278 | 89 2 | 3.14679 28441 | 87 |
| 0.99668 08734 | 2.52058 82420 | 0.04238 14278 | 88 42 | 3.11062 28114 | 86 |
| 0.99481 79213 | 2.51624 57960 | 0.05295 96662 | 88 22 | 3.07445 27787 | 85 |
| 0.99254 49353 | 2.51095 04254 | 0.06352 63677 | 88 2 | 3.03828 27460 | 84 |
| 0.98986 42745 | 2.50470 93354 | 0.07407 90993 | 87 42 | 3.00211 27133 | 83 |
| 0.98677 87139 | 2.49753 10120 | 0.08461 53590 | 87 22 | 2.96594 26806 | 82 |
| 0.98329 14382 | 2.48942 52067 | 0.09513 25631 | 87 2 | 2.92977 26479 | 81 |
| 0.97940 60344 | 2.48040 29203 | 0.10562 80337 | 86 41 | 2.89360 26152 | 80 |
| 0.97512 64836 | 2.47047 63835 | 0.11609 89854 | 86 20 | 2.85743 25825 | 79 |
| 0.97045 71520 | 2.45965 90364 | 0.12654 25123 | 85 59 | 2.82126 25499 | 78 |
| 0.96540 27806 | 2.44796 55051 | 0.13695 55734 | 85 38 | 2.78509 25172 | 77 |
| 0.95996 84748 | 2.43541 15773 | 0.14733 49785 | 85 16 | 2.74892 24845 | 76 |
| 0.95415 96925 | 2.42201 41749 | 0.15767 73727 | 84 54 | 2.71275 24518 | 75 |
| 0.94798 22318 | 2.40779 13262 | 0.16797 92208 | 84 32 | 2.67658 24191 | 74 |
| 0.94144 22181 | 2.39276 21349 | 0.17823 67907 | 84 9 | 2.64041 23864 | 73 |
| 0.93454 60898 | 2.37694 67487 | 0.18844 61360 | 83 45 | 2.60424 23537 | 72 |
| 0.92730 05843 | 2.36036 63252 | 0.19860 30778 | 83 21 | 2.56807 23210 | 71 |
| 0.91971 27230 | 2.34304 29976 | 0.20870 31860 | 82 57 | 2.53190 22883 | 70 |
| 0.91178 97950 | 2.32499 98377 | 0.21874 17592 | 82 32 | 2.49573 22556 | 69 |
| 0.90353 93417 | 2.30626 08184 | 0.22871 38038 | 82 7 | 2.45956 22230 | 68 |
| 0.89496 91397 | 2.28685 07750 | 0.23861 40125 | 81 41 | 2.42339 21903 | 67 |
| 0.88608 71836 | 2.26679 53647 | 0.24843 67407 | 81 14 | 2.38722 21576 | 66 |
| 0.87690 16690 | 2.24612 10260 | 0.25817 59833 | 80 47 | 2.35105 21249 | 65 |
| 0.86742 09743 | 2.22485 49364 | 0.26782 53494 | 80 19 | 2.31488 20922 | 64 |
| 0.85765 36425 | 2.20302 49697 | 0.27737 80358 | 79 50 | 2.27871 20595 | 63 |
| 0.84760 83633 | 2.18065 96524 | 0.28682 68004 | 79 20 | 2.24254 20268 | 62 |
| 0.83729 39541 | 2.15778 81197 | 0.29616 39332 | 78 50 | 2.20637 19941 | 61 |
| 0.82671 93416 | 2.13444 00706 | 0.30538 12272 | 78 19 | 2.17020 19614 | 60 |
| 0.81589 35429 | 2.11064 57227 | 0.31446 99478 | 77 47 | 2.13403 19287 | 59 |
| 0.80482 56467 | 2.08643 57672 | 0.32342 08014 | 77 14 | 2.09786 18960 | 58 |
| 0.79352 47945 | 2.06184 13229 | 0.33222 39026 | 76 40 | 2.06169 18634 | 57 |
| 0.78200 01623 | 2.03689 38902 | 0.34086 87415 | 76 5 | 2.02552 18307 | 56 |
| 0.77026 09411 | 2.01162 53056 | 0.34934 41494 | 75 29 | 1.98935 17980 | 55 |
| 0.75831 63194 | 1.98606 76958 | 0.35763 82644 | 74 53 | 1.95318 17653 | 54 |
| 0.74617 54642 | 1.96025 34320 | 0.36573 84971 | 74 14 | 1.91701 17326 | 53 |
| 0.73384 75039 | 1.93421 50843 | 0.37363 14953 | 73 35 | 1.88084 16999 | 52 |
| 0.72134 15096 | 1.90798 53771 | 0.38130 31100 | 72 55 | 1.84467 16672 | 51 |
| 0.70866 64787 | 1.88159 71433 | 0.38873 83616 | 72 13 | 1.80850 16345 | 50 |
| 0.69583 13178 | 1.85508 32817 | 0.39592 14068 | 71 30 | 1.77233 16018 | 49 |
| 0.68284 48256 | 1.82847 67117 | 0.40283 55079 | 70 46 | 1.73616 15691 | 48 |
| 0.66971 56781 | 1.80181 03311 | 0.40946 30040 | 70 1 | 1.69999 15365 | 47 |
| 0.65645 24120 | 1.77511 69734 | 0.41578 52846 | 69 14 | 1.66382 15038 | 46 |
| 0.64306 34108 | 1.74842 93662 | 0.42178 27675 | 68 25 | 1.62765 14711 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 3.3698680267, K′ = 1.5784865777, E = 1.027843620, E′ = 1.5629622295,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.03744 29781 | 2 9 | 0.02600 53438 | 1.00048 71379 | 0.01396 87846 |
| 2 | 0.07488 59561 | 4 17 | 0.05190 80180 | 1.00194 80481 | 0.02793 96081 |
| 3 | 0.11232 89342 | 6 26 | 0.07760 64875 | 1.00438 12208 | 0.04191 44920 |
| 4 | 0.14977 19123 | 8 35 | 0.10300 14601 | 1.00778 41400 | 0.05589 54231 |
| 5 | 0.18721 48904 | 10 40 | 0.12799 69416 | 1.01215 32844 | 0.06988 43359 |
| 6 | 0.22465 78684 | 12 46 | 0.15250 12188 | 1.01748 41292 | 0.08388 30956 |
| 7 | 0.26210 08465 | 14 51 | 0.17642 77402 | 1.02377 11470 | 0.09789 34813 |
| 8 | 0.29954 38246 | 16 55 | 0.19969 58914 | 1.03100 78103 | 0.11191 71690 |
| 9 | 0.33698 68027 | 18 58 | 0.22223 16400 | 1.03918 65941 | 0.12595 57152 |
| 10 | 0.37442 97807 | 20 59 | 0.24396 80481 | 1.04829 89781 | 0.14001 05412 |
| 11 | 0.41187 27588 | 22 58 | 0.26484 56468 | 1.05833 54510 | 0.15408 29167 |
| 12 | 0.44931 57369 | 24 56 | 0.28481 26740 | 1.06928 55135 | 0.16817 39451 |
| 13 | 0.48675 87150 | 26 52 | 0.30382 51779 | 1.08113 76835 | 0.18228 45483 |
| 14 | 0.52420 16930 | 28 46 | 0.32184 69961 | 1.09387 95005 | 0.19641 54524 |
| 15 | 0.56164 46711 | 30 38 | 0.33884 96193 | 1.10749 75312 | 0.21056 71740 |
| 16 | 0.59908 76492 | 32 28 | 0.35481 19530 | 1.12197 73762 | 0.22474 00071 |
| 17 | 0.63653 06273 | 34 16 | 0.36971 99918 | 1.13730 36763 | 0.23893 40100 |
| 18 | 0.67397 36053 | 36 2 | 0.38356 64197 | 1.15346 01207 | 0.25314 89941 |
| 19 | 0.71141 65834 | 37 46 | 0.39635 01539 | 1.17042 94549 | 0.26738 45123 |
| 20 | 0.74885 95615 | 39 27 | 0.40807 58450 | 1.18819 34902 | 0.28163 98484 |
| 21 | 0.78630 25396 | 41 6 | 0.41875 33497 | 1.20673 31139 | 0.29591 40077 |
| 22 | 0.82374 55176 | 42 42 | 0.42839 71871 | 1.22602 82998 | 0.31020 57076 |
| 23 | 0.86118 84957 | 44 16 | 0.43702 59916 | 1.24605 81209 | 0.32451 33701 |
| 24 | 0.89863 14738 | 45 48 | 0.44466 19725 | 1.26680 07616 | 0.33883 51142 |
| 25 | 0.93607 44519 | 47 18 | 0.45133 03888 | 1.28823 35321 | 0.35316 87494 |
| 26 | 0.97351 74299 | 48 45 | 0.45705 90462 | 1.31033 28836 | 0.36751 17704 |
| 27 | 1.01096 04080 | 50 10 | 0.46187 78212 | 1.33307 44242 | 0.38186 13526 |
| 28 | 1.04840 33861 | 51 32 | 0.46581 82181 | 1.35643 29365 | 0.39621 43484 |
| 29 | 1.08584 63641 | 52 52 | 0.46891 29597 | 1.38038 23962 | 0.41056 72843 |
| 30 | 1.12328 93422 | 54 10 | 0.47119 56148 | 1.40489 59917 | 0.42491 63594 |
| 31 | 1.16073 23203 | 55 26 | 0.47270 02620 | 1.42994 61457 | 0.43925 74448 |
| 32 | 1.19817 52984 | 56 39 | 0.47346 11908 | 1.45550 45373 | 0.45358 60835 |
| 33 | 1.23561 82764 | 57 50 | 0.47351 26377 | 1.48154 21259 | 0.46789 74917 |
| 34 | 1.27306 12545 | 59 0 | 0.47288 85574 | 1.50802 91764 | 0.48218 65611 |
| 35 | 1.31050 42326 | 60 7 | 0.47162 24256 | 1.53493 52855 | 0.49644 78621 |
| 36 | 1.34794 72107 | 61 12 | 0.46974 70729 | 1.56222 94100 | 0.51067 56480 |
| 37 | 1.38539 01887 | 62 15 | 0.46729 45464 | 1.58987 98960 | 0.52486 38600 |
| 38 | 1.42283 31668 | 63 16 | 0.46429 59969 | 1.61785 45092 | 0.53900 61335 |
| 39 | 1.46027 61449 | 64 15 | 0.46078 15892 | 1.64612 04680 | 0.55309 58052 |
| 40 | 1.49771 91230 | 65 12 | 0.45678 04338 | 1.67464 44762 | 0.56712 59210 |
| 41 | 1.53516 21010 | 66 7 | 0.45232 05363 | 1.70339 27583 | 0.58108 92454 |
| 42 | 1.57260 50791 | 67 1 | 0.44742 87637 | 1.73233 10960 | 0.59497 82708 |
| 43 | 1.61004 80572 | 67 53 | 0.44213 08242 | 1.76142 48657 | 0.60878 52287 |
| 44 | 1.64749 10353 | 68 44 | 0.43645 12599 | 1.79063 90777 | 0.62250 21016 |
| 45 | 1.68493 40133 | 69 32 | 0.43041 34495 | 1.81993 84164 | 0.63612 06349 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.229567159881194$, $00 = 0.5464169465$, $HK = 1.4575481002$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90−r |
|---|---|---|---|---|---|
| 1.00000 00000 | 2.68054 03437 | 0.00000 00000 | 90° 0′ | 3.36986 80267 | 90 |
| 0.99978 62112 | 2.68000 36787 | 0.01069 49135 | 89 42 | 3.33242 50486 | 89 |
| 0.99914 50809 | 2.67839 44283 | 0.02138 78301 | 89 24 | 3.29498 20705 | 88 |
| 0.99807 73170 | 2.67571 48255 | 0.03207 67423 | 89 6 | 3.25753 90925 | 87 |
| 0.99658 40972 | 2.67196 85860 | 0.04275 96209 | 88 48 | 3.22009 61144 | 86 |
| 0.99466 70666 | 2.66716 09043 | 0.05343 44040 | 88 30 | 3.18265 31363 | 85 |
| 0.99232 83334 | 2.66129 84418 | 0.06409 89867 | 88 12 | 3.14521 01582 | 84 |
| 0.98957 04645 | 2.65438 93156 | 0.07475 12085 | 87 53 | 3.10776 71802 | 83 |
| 0.98639 64786 | 2.64644 30842 | 0.08538 88428 | 87 35 | 3.07032 42021 | 82 |
| 0.98280 98400 | 2.63747 07296 | 0.09600 95847 | 87 16 | 3.03288 12240 | 81 |
| 0.97881 44497 | 2.62748 46381 | 0.10661 10385 | 86 57 | 2.99543 82459 | 80 |
| 0.97441 46367 | 2.61649 85778 | 0.11719 07054 | 86 37 | 2.95799 52679 | 79 |
| 0.96961 51474 | 2.60452 76741 | 0 12774 59701 | 86 18 | 2.92055 22898 | 78 |
| 0.96442 11348 | 2.59158 83828 | 0 13827 40870 | 85 58 | 2.88310 93117 | 77 |
| 0.95883 81466 | 2.57769 84606 | 0.14877 21662 | 85 38 | 2.84566 63336 | 76 |
| 0.95287 21117 | 2.56287 69342 | 0.15923 71580 | 85 17 | 2.80822 33556 | 75 |
| 0.94652 93269 | 2.54714 40664 | 0.16966 58376 | 84 56 | 2.77078 03775 | 74 |
| 0.93981 64421 | 2.53052 13208 | 0.18005 47885 | 84 35 | 2.73333 73994 | 73 |
| 0.93274 04449 | 2.51303 13248 | 0.19040 03849 | 84 13 | 2.69589 44213 | 72 |
| 0.92530 86446 | 2.49469 78294 | 0.20069 87739 | 83 51 | 2.65845 14433 | 71 |
| 0.91752 86553 | 2.47554 56695 | 0.21094 58556 | 83 28 | 2.62100 84652 | 70 |
| 0.90940 83786 | 2.45560 07207 | 0.22113 72633 | 83 5 | 2.58356 54871 | 69 |
| 0.90095 59853 | 2.43488 98556 | 0.23126 83422 | 82 41 | 2.54612 25090 | 68 |
| 0.89217 98975 | 2.41344 08985 | 0.24133 41265 | 82 16 | 2.50867 95310 | 67 |
| 0.88308 87690 | 2.39128 25787 | 0.25132 93157 | 81 51 | 2.47123 65529 | 66 |
| 0.87369 14660 | 2.36844 44831 | 0.26124 82501 | 81 25 | 2.43379 35748 | 65 |
| 0.86399 70475 | 2.34495 70070 | 0.27108 48837 | 80 59 | 2.39635 05967 | 64 |
| 0 85401 47452 | 2.32085 13053 | 0.28083 27574 | 80 32 | 2.35890 76187 | 63 |
| 0 84375 39427 | 2.29615 92414 | 0.29048 49692 | 80 4 | 2.32146 46406 | 62 |
| 0.83322 41555 | 2.27091 33365 | 0.30003 41444 | 79 35 | 2.28402 16625 | 61 |
| 0.82243 50100 | 2.24514 67182 | 0.30947 24031 | 79 5 | 2.24657 86844 | 60 |
| 0.81139 62227 | 2.21889 30687 | 0.31879 13276 | 78 35 | 2.20913 57064 | 59 |
| 0.80011 75795 | 2.19218 65719 | 0.32798 19272 | 78 4 | 2.17169 27283 | 58 |
| 0.78860 89149 | 2.16506 18621 | 0.33703 46027 | 77 31 | 2.13424 97502 | 57 |
| 0.77688 00911 | 2.13755 39706 | 0.34593 91087 | 76 58 | 2.09680 67721 | 56 |
| 0.76494 09778 | 2.10969 82742 | 0.35468 45152 | 76 23 | 2.05936 37941 | 55 |
| 0.75280 14315 | 2.08153 04423 | 0.36325 91686 | 75 48 | 2.02192 08160 | 54 |
| 0.74047 12755 | 2.05308 63856 | 0.37165 06505 | 75 11 | 1.98447 78379 | 53 |
| 0.72796 02805 | 2.02440 22044 | 0.37984 57377 | 74 34 | 1.94703 48599 | 52 |
| 0.71527 81443 | 1.99551 41373 | 0.38783 03601 | 73 55 | 1.90959 18818 | 51 |
| 0.70243 44736 | 1.96645 85115 | 0.39558 95596 | 73 14 | 1.87214 89037 | 50 |
| 0 68943 87648 | 1.93727 16923 | 0.40310 74491 | 72 33 | 1.83470 59256 | 49 |
| 0.67630 03866 | 1.90799 00345 | 0.41036 71725 | 71 50 | 1.79726 29476 | 48 |
| 0.66302 85617 | 1.87864 98345 | 0.41735 08655 | 71 6 | 1.75981 99695 | 47 |
| 0.64963 23506 | 1.84928 72824 | 0.42403 96200 | 70 20 | 1.72237 69914 | 46 |
| 0.63612 06349 | 1.81993 84164 | 0.43041 34495 | 69 32 | 1.68493 40133 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 3.5004224992, K′ = 1.5766779816, E = 1.022312588, E′ = 1.5649475630,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 90000 | 0.00000 00000 |
| 1 | 0.03889 35833 | 2 14 | 0.02751 52459 | 1.00053 54142 | 0.01357 81428 |
| 2 | 0.07778 71666 | 4 27 | 0.05491 49171 | 1.00214 11230 | 0.02715 91294 |
| 3 | 0.11668 07500 | 6 40 | 0.08208 48196 | 1.00481 55243 | 0.04074 57840 |
| 4 | 0.15557 43333 | 8 53 | 0.10891 34862 | 1.00855 59486 | 0.05434 08922 |
| 5 | 0.19446 79166 | 11 4 | 0.13529 34531 | 1.01335 86590 | 0.06794 71815 |
| 6 | 0.23336 14999 | 13 15 | 0.16112 24388 | 1.01921 88518 | 0.08156 73027 |
| 7 | 0.27225 50833 | 15 25 | 0.18630 43989 | 1.02613 06577 | 0.09520 38101 |
| 8 | 0.31114 86666 | 17 33 | 0.21075 04315 | 1.03408 71422 | 0.10885 91438 |
| 9 | 0.35004 22499 | 19 40 | 0.23437 95237 | 1.04308 03072 | 0.12253 56111 |
| 10 | 0.38893 58332 | 21 45 | 0.25711 91248 | 1.05310 10924 | 0.13623 53681 |
| 11 | 0.42782 94166 | 23 48 | 0.27890 55463 | 1.06413 93774 | 0.14996 04030 |
| 12 | 0.46672 29999 | 25 50 | 0.29968 41874 | 1.07618 39836 | 0.16371 25182 |
| 13 | 0.50561 65832 | 27 50 | 0.31940 95974 | 1.08922 26769 | 0.17749 33141 |
| 14 | 0.54451 01665 | 29 47 | 0.33804 53836 | 1.10324 21710 | 0.19130 41733 |
| 15 | 0.58340 37499 | 31 42 | 0.35556 39822 | 1.11822 81308 | 0.20514 62446 |
| 16 | 0 62229 73332 | 33 35 | 0.37194 63079 | 1.13416 51764 | 0.21902 04287 |
| 17 | 0.66119 09165 | 35 26 | 0.38718 13038 | 1.15103 68883 | 0.23292 73637 |
| 18 | 0.70008 44998 | 37 14 | 0.40126 54102 | 1.16882 58124 | 0.24686 74120 |
| 19 | 0.73897 80832 | 38 59 | 0.41420 19722 | 1.18751 34668 | 0.26084 06476 |
| 20 | 0.77787 16665 | 40 42 | 0.42600 06064 | 1.20708 03483 | 0.27484 68440 |
| 21 | 0.81676 52498 | 42 23 | 0.43667 65427 | 1.22750 59404 | 0.28888 54637 |
| 22 | 0.85565 88331 | 44 1 | 0.44624 99581 | 1.24876 87226 | 0.30295 56475 |
| 23 | 0.89455 24165 | 45 37 | 0.45474 53170 | 1.27084 61798 | 0.31705 62057 |
| 24 | 0.93344 59998 | 47 10 | 0.46219 07281 | 1.29371 48135 | 0.33118 56095 |
| 25 | 0.97233 95831 | 48 40 | 0.46861 73287 | 1.31735 01537 | 0.34534 19839 |
| 26 | 1.01123 31664 | 50 8 | 0.47405 87042 | 1.34172 67728 | 0.35952 31012 |
| 27 | 1.05012 67498 | 51 33 | 0.47855 03463 | 1.36681 82994 | 0.37372 63757 |
| 28 | 1.08902 03331 | 52 56 | 0.48212 91569 | 1.39259 74348 | 0.38794 88593 |
| 29 | 1.12791 39164 | 54 17 | 0.48483 29959 | 1.41903 59703 | 0.40218 72381 |
| 30 | 1.16680 74997 | 55 35 | 0.48670 02770 | 1.44610 48057 | 0.41643 78306 |
| 31 | 1.20570 10830 | 56 50 | 0.48776 96093 | 1.47377 39701 | 0.43069 65861 |
| 32 | 1.24459 46664 | 58 4 | 0.48807 94838 | 1.50201 26433 | 0.44495 90849 |
| 33 | 1.28348 82497 | 59 14 | 0.48766 80032 | 1.53078 91792 | 0.45922 05390 |
| 34 | 1.32238 18330 | 60 23 | 0.48657 26520 | 1.56007 11317 | 0.47347 57948 |
| 35 | 1.36127 54163 | 61 30 | 0.48483 01039 | 1.58982 52804 | 0.48771 93356 |
| 36 | 1.40016 89997 | 62 34 | 0.48247 60647 | 1.62001 76598 | 0.50194 52865 |
| 37 | 1.43906 25830 | 63 36 | 0.47954 51456 | 1.65061 35895 | 0.51614 74196 |
| 38 | 1.47795 61663 | 64 36 | 0.47607 07644 | 1.68157 77058 | 0.53031 91603 |
| 39 | 1.51684 97496 | 65 35 | 0.47208 50753 | 1.71287 39955 | 0.54445 35952 |
| 40 | 1.55574 33330 | 66 31 | 0.46761 89121 | 1.74446 58318 | 0.55854 34803 |
| 41 | 1.59463 69163 | 67 25 | 0.46270 17621 | 1.77631 60110 | 0.57258 12511 |
| 42 | 1.63353 04996 | 68 18 | 0.45736 17475 | 1.80838 67918 | 0.58655 90333 |
| 43 | 1.67242 40829 | 69 9 | 0.45162 56249 | 1.84063 99362 | 0.60046 86540 |
| 44 | 1.71131 76663 | 69 58 | 0.44551 87962 | 1.87303 67513 | 0.61430 16549 |
| 45 | 1.75021 12496 | 70 45 | 0.43906 53283 | 1.90553 81344 | 0.62804 93057 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.242912974306665$, $\Theta 0 = 0.5211317465$, $HK = 1.4872214813$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 2.86452 59727 | 0.00000 00000 | 90° 0′ | 3.50042 24992 | 90 |
| 0.99977 91249 | 2.86392 54580 | 0.01078 10889 | 89 44 | 3.46152 89158 | 89 |
| 0.99911 67583 | 2.86212 47652 | 0.02156 04536 | 89 27 | 3.42263 53325 | 88 |
| 0.99801 36755 | 2.85912 64461 | 0.03233 63597 | 89 11 | 3.38374 17492 | 87 |
| 0.99647 11670 | 2.85493 47485 | 0.04310 70526 | 88 55 | 3.34484 81659 | 86 |
| 0.99449 10345 | 2.84955 56077 | 0.05387 07471 | 88 38 | 3.30595 45826 | 85 |
| 0.99207 55874 | 2.84299 66356 | 0.06462 56168 | 88 21 | 3.26706 09992 | 84 |
| 0.98922 76367 | 2.83526 71062 | 0.07536 97836 | 88 5 | 3.22816 74159 | 83 |
| 0.98595 04884 | 2.82637 79377 | 0.08610 13069 | 87 48 | 3.18927 38326 | 82 |
| 0.98224 79350 | 2.81634 16722 | 0.09681 81718 | 87 30 | 3.15038 02493 | 81 |
| 0.97812 42473 | 2.80517 24517 | 0.10751 82779 | 87 13 | 3.11148 66659 | 80 |
| 0.97358 41628 | 2.79288 59919 | 0.11819 94268 | 86 55 | 3.07259 30826 | 79 |
| 0.96863 28755 | 2.77949 95523 | 0.12885 93097 | 86 37 | 3.03369 94993 | 78 |
| 0.96327 60226 | 2.76503 19042 | 0.13949 54938 | 86 19 | 2.99480 59160 | 77 |
| 0.95751 96711 | 2.74950 32957 | 0.15010 54088 | 86 1 | 2.95591 23326 | 76 |
| 0.95137 03036 | 2.73293 54142 | 0.16068 63318 | 85 42 | 2.91701 87493 | 75 |
| 0.94483 48022 | 2.71535 13465 | 0.17123 53724 | 85 23 | 2.87812 51660 | 74 |
| 0.93792 04329 | 2.69677 55363 | 0.18174 94560 | 85 3 | 2.83923 15827 | 73 |
| 0.93063 48276 | 2.67723 37397 | 0.19222 53067 | 84 43 | 2.80033 79993 | 72 |
| 0.92298 59663 | 2.65675 29786 | 0.20265 94294 | 84 22 | 2.76144 44160 | 71 |
| 0.91498 21585 | 2.63536 14921 | 0.21304 80901 | 84 1 | 2.72255 08327 | 70 |
| 0.90663 20234 | 2.61308 86858 | 0.22338 72956 | 83 39 | 2.68365 72494 | 69 |
| 0.89794 44698 | 2.58996 50797 | 0.23367 27719 | 83 17 | 2.64476 36660 | 68 |
| 0.88892 86753 | 2.56602 22548 | 0.24389 99414 | 82 54 | 2.60587 00827 | 67 |
| 0.87959 40653 | 2.54129 27973 | 0.25406 38981 | 82 31 | 2.56697 64994 | 66 |
| 0.86995 02909 | 2.51581 02430 | 0.26415 93822 | 82 7 | 2.52808 29161 | 65 |
| 0.86000 72069 | 2.48960 90190 | 0.27418 07525 | 81 42 | 2.48918 93327 | 64 |
| 0.84977 48495 | 2.46272 43859 | 0.28412 19576 | 81 16 | 2.45029 57494 | 63 |
| 0.83926 34134 | 2.43519 23782 | 0.29397 65053 | 80 50 | 2.41140 21661 | 62 |
| 0.82848 32287 | 2.40704 97447 | 0.30373 74301 | 80 23 | 2.37250 85828 | 61 |
| 0.81744 47382 | 2.37833 38874 | 0.31339 72593 | 79 55 | 2.33361 49994 | 60 |
| 0.80615 84738 | 2.34908 28015 | 0.32294 79773 | 79 26 | 2.29472 14161 | 59 |
| 0.79463 50337 | 2.31933 50143 | 0.33238 09873 | 78 56 | 2.25582 78328 | 58 |
| 0.78288 50590 | 2.28912 95239 | 0.34168 70724 | 78 26 | 2.21693 42495 | 57 |
| 0.77091 92109 | 2.25850 57383 | 0.35085 63539 | 77 54 | 2.17804 06662 | 56 |
| 0.75874 81476 | 2.22750 34151 | 0.35987 82486 | 77 21 | 2.13914 70828 | 55 |
| 0.74638 25018 | 2.19616 26008 | 0.36874 14237 | 76 47 | 2.10025 34995 | 54 |
| 0.73383 28587 | 2.16452 35708 | 0.37743 37507 | 76 12 | 2.06135 99162 | 53 |
| 0.72110 97334 | 2.13262 67708 | 0.38594 22578 | 75 36 | 2.02246 63329 | 52 |
| 0.70822 35503 | 2.10051 27578 | 0.39425 30813 | 74 58 | 1.98357 27495 | 51 |
| 0.69518 46210 | 2.06822 21426 | 0.40235 14155 | 74 20 | 1.94467 91662 | 50 |
| 0.68200 31247 | 2.03579 55331 | 0.41022 14630 | 73 40 | 1.90578 55829 | 49 |
| 0.66868 90878 | 2.00327 34790 | 0.41784 63843 | 72 58 | 1.86689 19996 | 48 |
| 0.65525 23646 | 1.97069 64170 | 0.42520 82479 | 72 16 | 1.82799 84162 | 47 |
| 0.64170 26188 | 1.93810 46179 | 0.43228 79822 | 71 31 | 1.78910 48329 | 46 |
| 0.62804 93057 | 1.90553 81344 | 0.43906 53283 | 70 45 | 1.75021 12496 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 3.6518559695, K′ = 1.5751136078, E = 1.017236918, E′ = 1.5664967878,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.04057 61774 | 2 1 | 0.02925 15342 | 1.00059 38572 | 0.01311 92586 |
| 2 | 0.08115 23549 | 4 29 | 0.05837 13484 | 1.00237 48641 | 0.02624 22974 |
| 3 | 0.12172 85323 | 6 55 | 0.08722 94380 | 1.00534 13262 | 0.03937 28749 |
| 4 | 0.16230 47098 | 9 16 | 0.11569 91812 | 1.00949 04192 | 0.05251 47063 |
| 5 | 0.20288 08872 | 11 33 | 0.14365 89152 | 1.01481 81886 | 0.06567 14426 |
| 6 | 0.24345 70646 | 13 49 | 0.17099 33783 | 1.02131 95491 | 0.07884 66485 |
| 7 | 0.28403 32421 | 16 4 | 0.19759 49853 | 1.02898 82841 | 0.09204 37819 |
| 8 | 0.32460 94195 | 18 17 | 0.22336 49075 | 1.03781 70450 | 0.10526 61731 |
| 9 | 0.36518 55969 | 20 29 | 0.24821 39381 | 1.04779 73504 | 0.11851 70041 |
| 10 | 0.40576 17744 | 22 39 | 0.27206 31341 | 1.05891 95857 | 0.13179 92889 |
| 11 | 0.44633 79518 | 24 46 | 0.29484 42309 | 1.07117 30024 | 0.14511 58534 |
| 12 | 0.48691 41293 | 26 52 | 0.31649 98365 | 1.08454 57174 | 0.15846 93168 |
| 13 | 0.52749 03067 | 28 56 | 0.33698 34175 | 1.09902 47131 | 0.17186 20726 |
| 14 | 0.56806 64841 | 30 58 | 0.35625 90959 | 1.11459 58374 | 0.18529 62711 |
| 15 | 0.60864 26616 | 32 55 | 0.37430 12782 | 1.13124 38038 | 0.19877 38016 |
| 16 | 0.64921 88390 | 34 51 | 0.39109 41430 | 1.14895 21925 | 0.21229 62758 |
| 17 | 0.68979 50165 | 36 44 | 0.40663 10147 | 1.16770 34514 | 0.22586 50123 |
| 18 | 0.73037 11939 | 38 36 | 0.42091 36481 | 1.18747 88983 | 0.23948 10211 |
| 19 | 0.77094 73713 | 40 24 | 0.43395 14533 | 1.20825 87235 | 0.25314 49894 |
| 20 | 0.81152 35488 | 42 9 | 0.44576 06829 | 1.23002 19929 | 0.26685 72683 |
| 21 | 0.85209 97262 | 43 51 | 0.45636 36044 | 1.25274 66524 | 0.28061 78600 |
| 22 | 0.89267 59037 | 45 31 | 0.46578 76783 | 1.27640 95335 | 0.29442 64067 |
| 23 | 0.93325 20811 | 47 8 | 0.47406 47564 | 1.30098 63590 | 0.30828 21794 |
| 24 | 0.97382 82585 | 48 42 | 0.48123 03147 | 1.32645 17509 | 0.32218 40690 |
| 25 | 1.01440 44360 | 50 13 | 0.48732 27312 | 1.35277 92393 | 0.33613 05773 |
| 26 | 1.05498 06134 | 51 42 | 0.49238 26159 | 1.37994 12721 | 0.35011 98097 |
| 27 | 1.09555 67908 | 53 8 | 0.49645 21966 | 1.40790 92268 | 0.36414 94689 |
| 28 | 1.13613 29683 | 54 31 | 0.49957 47663 | 1.43665 34239 | 0.37821 68497 |
| 29 | 1.17670 91457 | 55 51 | 0.50179 41897 | 1.46614 31412 | 0.39231 88350 |
| 30 | 1.21728 53232 | 57 9 | 0.50315 44701 | 1.49634 66307 | 0.40645 18927 |
| 31 | 1.25786 15006 | 58 25 | 0.50369 93739 | 1.52723 11369 | 0.42061 20743 |
| 32 | 1.29843 76780 | 59 38 | 0.50347 21104 | 1.55876 29167 | 0.43479 50141 |
| 33 | 1.33901 38555 | 60 48 | 0.50251 50624 | 1.59090 72622 | 0.44899 59303 |
| 34 | 1.37959 00329 | 61 56 | 0.50086 95651 | 1.62362 85241 | 0.46320 96265 |
| 35 | 1.42016 62104 | 63 2 | 0.49857 57270 | 1.65689 01387 | 0.47743 04952 |
| 36 | 1.46074 23878 | 64 5 | 0.49567 22903 | 1.69065 46558 | 0.49165 25218 |
| 37 | 1.50131 85652 | 65 7 | 0.49219 65260 | 1.72488 37696 | 0.50586 92908 |
| 38 | 1.54189 47427 | 66 6 | 0.48818 41583 | 1.75953 83514 | 0.52007 39919 |
| 39 | 1.58247 09201 | 67 3 | 0.48366 93168 | 1.79457 84847 | 0.53425 94285 |
| 40 | 1.62304 70975 | 67 58 | 0.47868 45099 | 1.82996 35024 | 0.54841 80268 |
| 41 | 1.66362 32750 | 68 51 | 0.47326 06189 | 1.86565 20265 | 0.56254 18461 |
| 42 | 1.70419 94524 | 69 42 | 0.46742 69071 | 1.90160 20099 | 0.57662 25903 |
| 43 | 1.74477 56299 | 70 31 | 0.46121 10428 | 1.93777 07807 | 0.59065 16209 |
| 44 | 1.78535 18073 | 71 19 | 0.45463 91336 | 1.97411 50881 | 0.60461 99704 |
| 45 | 1.82592 79847 | 72 5 | 0.44773 57684 | 2.01059 11517 | 0.61851 83573 |
| 90−r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.257940195766337$, $\Theta 0 = 0.4929628191$, $HK = 1.5205617314$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 3.09301 99213 | 0.00000 00000 | 90° 0′ | 3.65185 59695 | 90 |
| 0.99977 07150 | 3.09233 85676 | 0.01085 90483 | 89 45 | 3.61127 97920 | 89 |
| 0.99908 31458 | 3.09029 54977 | 0.02171 66503 | 89 31 | 3.57070 36146 | 88 |
| 0.99793 81489 | 3.08689 36827 | 0.03257 13506 | 89 16 | 3.53012 74372 | 87 |
| 0.99633 71496 | 3.08213 80679 | 0.04342 16747 | 89 1 | 3.48955 12597 | 86 |
| 0.99428 21381 | 3.07603 55627 | 0.05426 61204 | 88 47 | 3.44897 50823 | 85 |
| 0.99177 56649 | 3.06859 50269 | 0.06510 31473 | 88 32 | 3.40839 89048 | 84 |
| 0.98882 08340 | 3.05982 72527 | 0.07593 11673 | 88 17 | 3.36782 27274 | 83 |
| 0.98542 12955 | 3.04974 49431 | 0.08674 85345 | 88 2 | 3.32724 65500 | 82 |
| 0.98158 12363 | 3.03836 26866 | 0.09755 35344 | 87 46 | 3.28667 03725 | 81 |
| 0.97730 53698 | 3.02569 69280 | 0.10834 43731 | 87 30 | 3.24609 41951 | 80 |
| 0.97259 89240 | 3.01176 59358 | 0.11911 91660 | 87 14 | 3.20551 80177 | 79 |
| 0.96746 76286 | 2.99658 97659 | 0 12987 59255 | 86 58 | 3.16494 18402 | 78 |
| 0.96191 77007 | 2.98019 02223 | 0.14061 25487 | 86 42 | 3.12436 56628 | 77 |
| 0.95595 58299 | 2.96259 08137 | 0.15132 68040 | 86 25 | 3.08378 94853 | 76 |
| 0.94958 91609 | 2.94381 67083 | 0.16201 63172 | 86 8 | 3.04321 33079 | 75 |
| 0.94282 52769 | 2.92389 46843 | 0.17267 85562 | 85 50 | 3.00263 71305 | 74 |
| 0.93567 21802 | 2.90285 30783 | 0.18331 08161 | 85 32 | 2.96206 09530 | 73 |
| 0.92813 82732 | 2.88072 17308 | 0.19391 02013 | 85 14 | 2.92148 47756 | 72 |
| 0.92023 23376 | 2.85753 19293 | 0.20447 36088 | 84 55 | 2.88090 85981 | 71 |
| 0.91196 35133 | 2.83331 63492 | 0.21499 77081 | 84 36 | 2.84033 24207 | 70 |
| 0.90334 12763 | 2.80810 89917 | 0.22547 89218 | 84 16 | 2.79975 62433 | 69 |
| 0.89437 54154 | 2.78194 51210 | 0.23591 34034 | 83 55 | 2.75918 00658 | 68 |
| 0.88507 60096 | 2.75486 11988 | 0.24629 70143 | 83 34 | 2.71860 38884 | 67 |
| 0.87545 34034 | 2.72689 48173 | 0.25662 52995 | 83 13 | 2.67802 77109 | 66 |
| 0.86551 81826 | 2.69808 46313 | 0.26689 34606 | 82 51 | 2.63745 15335 | 65 |
| 0.85528 11491 | 2.66847 02880 | 0.27709 63287 | 82 28 | 2.59687 53561 | 64 |
| 0.84475 32958 | 2.63809 23575 | 0.28722 83335 | 82 4 | 2.55629 91786 | 63 |
| 0.83394 57809 | 2.60699 22604 | 0.29728 34722 | 81 39 | 2.51572 30012 | 62 |
| 0.82286 99019 | 2.57521 21966 | 0.30725 52753 | 81 14 | 2.47514 68238 | 61 |
| 0.81153 70701 | 2.54279 50725 | 0.31713 67705 | 80 48 | 2.43457 06463 | 60 |
| 0.79995 87840 | 2.50978 44281 | 0.32692 04449 | 80 21 | 2.39399 44689 | 59 |
| 0.78814 66036 | 2.47622 43648 | 0.33659 82039 | 79 53 | 2.35341 82914 | 58 |
| 0.77611 21247 | 2.44215 94723 | 0.34616 13287 | 79 24 | 2.31284 21140 | 57 |
| 0.76386 69524 | 2.40763 47564 | 0.35560 04313 | 78 54 | 2.27226 59366 | 56 |
| 0.75142 26764 | 2.37269 55671 | 0.36490 54063 | 78 23 | 2.23168 97591 | 55 |
| 0.73879 08451 | 2.33738 75276 | 0.37406 53814 | 77 51 | 2.19111 35817 | 54 |
| 0.72598 29409 | 2.30175 64635 | 0.38306 86651 | 77 18 | 2.15053 74042 | 53 |
| 0.71301 03561 | 2.26584 83337 | 0.39190 26919 | 76 44 | 2.10996 12268 | 52 |
| 0.69988 43682 | 2.22970 91619 | 0.40055 39659 | 76 8 | 2.06938 50494 | 51 |
| 0.68661 61172 | 2.19338 49695 | 0.40900 80023 | 75 31 | 2.02880 88719 | 50 |
| 0.67321 65825 | 2.15692 17102 | 0.41724 92673 | 74 53 | 1.98823 26945 | 49 |
| 0.65969 65607 | 2.12036 52053 | 0.42526 11165 | 74 13 | 1.94765 65171 | 48 |
| 0.64606 66446 | 2.08376 10820 | 0.43302 57335 | 73 32 | 1.90708 03396 | 47 |
| 0.63233 72022 | 2.04715 47117 | 0.44052 40667 | 72 49 | 1.86650 41622 | 46 |
| 0.61851 83573 | 2.01059 11517 | 0.44773 57684 | 72 5 | 1.82592 79847 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 3.8317419998, K′ = 1.5737921309, E = 1.0126635062, E′ = 1.5678090740,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.04257 49111 | 2 26 | 0.03129 75841 | 1.00066 67396 | 0.01256 98450 |
| 2 | 0.08514 98222 | 4 52 | 0.06244 25476 | 1.00266 63652 | 0.02514 45765 |
| 3 | 0.12772 47333 | 7 18 | 0.09328 44601 | 1.00599 70974 | 0.03772 90570 |
| 4 | 0.17029 96444 | 9 43 | 0.12367 72052 | 1.01065 59692 | 0.05032 81006 |
| 5 | 0.21287 45555 | 12 6 | 0.15348 09749 | 1.01663 88247 | 0.06294 64495 |
| 6 | 0.25544 94667 | 14 29 | 0.18256 40780 | 1.02394 03165 | 0.07558 87497 |
| 7 | 0.29802 43778 | 16 50 | 0.21080 45154 | 1.03255 39030 | 0.08825 95281 |
| 8 | 0.34059 92889 | 19 9 | 0.23809 12866 | 1.04247 18453 | 0.10096 31685 |
| 9 | 0.38317 42000 | 21 26 | 0.26432 54039 | 1.05368 52030 | 0.11370 38895 |
| 10 | 0.42574 91111 | 23 42 | 0.28942 06026 | 1.06618 38299 | 0.12648 57214 |
| 11 | 0.46832 40222 | 25 55 | 0.31330 37505 | 1.07995 63700 | 0.13931 24846 |
| 12 | 0.51089 89333 | 28 5 | 0.33591 49667 | 1.09499 02519 | 0.15218 77682 |
| 13 | 0.55347 38444 | 30 13 | 0.35720 74739 | 1.11127 16844 | 0.16511 49087 |
| 14 | 0.59604 87555 | 32 18 | 0.37714 72117 | 1.12878 56513 | 0.17809 69700 |
| 15 | 0.63862 36666 | 34 21 | 0.39571 22464 | 1.14751 59063 | 0.19113 67239 |
| 16 | 0.68119 85777 | 36 20 | 0.41289 20138 | 1.16744 49685 | 0.20423 66315 |
| 17 | 0.72377 34889 | 38 17 | 0.42868 64336 | 1.18855 41178 | 0.21739 88246 |
| 18 | 0.76634 84000 | 40 11 | 0.44310 49337 | 1.21082 33907 | 0.23062 50891 |
| 19 | 0.80892 33111 | 42 1 | 0.45616 54173 | 1.23423 15771 | 0.24391 68485 |
| 20 | 0.85149 82222 | 43 49 | 0.46789 32075 | 1.25875 62174 | 0.25727 51484 |
| 21 | 0.89407 31333 | 45 33 | 0.47831 99952 | 1.28437 36007 | 0.27070 06428 |
| 22 | 0.93664 80444 | 47 15 | 0.48748 28142 | 1.31105 87634 | 0.28419 35800 |
| 23 | 0.97922 29555 | 48 53 | 0.49542 30625 | 1.33878 54900 | 0.29775 37910 |
| 24 | 1.02179 78666 | 50 28 | 0.50218 55842 | 1.36752 63142 | 0.31138 06778 |
| 25 | 1.06437 27777 | 52 0 | 0.50781 78217 | 1.39725 25218 | 0.32507 32040 |
| 26 | 1.10694 76888 | 53 29 | 0.51236 90454 | 1.42793 41552 | 0.33882 98857 |
| 27 | 1.14952 25999 | 54 56 | 0.51588 96635 | 1.45954 00195 | 0.35264 87839 |
| 28 | 1.19209 75110 | 56 19 | 0.51843 06138 | 1.49203 76904 | 0.36652 74982 |
| 29 | 1.23467 24222 | 57 39 | 0.52004 28338 | 1.52539 35243 | 0.38046 31619 |
| 30 | 1.27724 73333 | 58 59 | 0.52077 68087 | 1.55957 26706 | 0.39445 24378 |
| 31 | 1.31982 22444 | 60 12 | 0.52068 21896 | 1.59453 90851 | 0.40849 15164 |
| 32 | 1.36239 71555 | 61 24 | 0.51980 74799 | 1.63025 55479 | 0.42257 61140 |
| 33 | 1.40497 20666 | 62 34 | 0.51819 97811 | 1.66668 36814 | 0.43670 14735 |
| 34 | 1.44754 69777 | 63 41 | 0.51590 45944 | 1.70378 39728 | 0.45086 23658 |
| 35 | 1.49012 18888 | 64 46 | 0.51296 56697 | 1.74151 57980 | 0.46505 30926 |
| 36 | 1.53269 67999 | 65 48 | 0.50942 48984 | 1.77983 74487 | 0.47926 74909 |
| 37 | 1.57527 17110 | 66 48 | 0.50532 22421 | 1.81870 61627 | 0.49349 89386 |
| 38 | 1.61784 66221 | 67 46 | 0.50069 56936 | 1.85807 81564 | 0.50774 03615 |
| 39 | 1.66042 15332 | 68 41 | 0.49558 12646 | 1.89790 86607 | 0.52198 42419 |
| 40 | 1.70299 64444 | 69 35 | 0.49001 29952 | 1.93815 19599 | 0.53622 26281 |
| 41 | 1.74557 13555 | 70 26 | 0.48402 29824 | 1.97876 14331 | 0.55044 71457 |
| 42 | 1.78814 62666 | 71 16 | 0.47764 14227 | 2.01968 95998 | 0.56464 90099 |
| 43 | 1.83072 11777 | 72 3 | 0.47089 66670 | 2.06088 81669 | 0.57881 90394 |
| 44 | 1.87329 60888 | 72 49 | 0.46381 52836 | 2.10230 80805 | 0.59294 76712 |
| 45 | 1.91587 09999 | 73 33 | 0.45642 21286 | 2.14389 95792 | 0.60702 49768 |
| 90-r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.275179804873563$, $\Theta 0 = 0.4610905222$, $HK = 1.5588714533$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 3.38728 70037 | 0.00000 00000 | 90° 0′ | 3.83174 19998 | 90 |
| 0.99976 05041 | 3.38649 90904 | 0.01092 82185 | 89 47 | 3.78916 70887 | 89 |
| 0.99904 23353 | 3.38413 65337 | 0.02185 52713 | 89 34 | 3.74659 21776 | 88 |
| 0.99784 64504 | 3.38020 28815 | 0.03277 99847 | 89 22 | 3.70401 72665 | 87 |
| 0.99617 44409 | 3.37470 40379 | 0.04370 11679 | 89 9 | 3.66144 23554 | 86 |
| 0.99402 85290 | 3.36764 82512 | 0.05461 76051 | 88 56 | 3.61886 74443 | 85 |
| 0.99141 15622 | 3.35904 60961 | 0.06552 80467 | 88 43 | 3.57629 25331 | 84 |
| 0.98832 70058 | 3.34891 04507 | 0.07643 12000 | 88 29 | 3.53371 76220 | 83 |
| 0.98477 89335 | 3.33725 64694 | 0.08732 57205 | 88 16 | 3.49114 27109 | 82 |
| 0.98077 20177 | 3.32410 15504 | 0.09821 02023 | 88 2 | 3.44856 77998 | 81 |
| 0.97631 15168 | 3.30946 52989 | 0.10908 31677 | 87 49 | 3.40599 28887 | 80 |
| 0.97140 32619 | 3.29336 94854 | 0.11994 30573 | 87 35 | 3.36341 79776 | 79 |
| 0.96605 36420 | 3.27583 79999 | 0.13078 82183 | 87 20 | 3.32084 30665 | 78 |
| 0.96026 95874 | 3.25689 68018 | 0.14161 68937 | 87 6 | 3.27826 81554 | 77 |
| 0.95405 85520 | 3.23657 38654 | 0.15242 72092 | 86 51 | 3.23569 32443 | 76 |
| 0.94742 84947 | 3.21489 91220 | 0.16321 71605 | 86 35 | 3.19311 83332 | 75 |
| 0.94038 78585 | 3.19190 43978 | 0.17398 45990 | 86 20 | 3.15054 34221 | 74 |
| 0.93294 55499 | 3.16762 33486 | 0.18472 72171 | 86 4 | 3.10796 85109 | 73 |
| 0.92511 09158 | 3.14209 13909 | 0.19544 25321 | 85 48 | 3.06539 35998 | 72 |
| 0.91689 37204 | 3.11534 56304 | 0.20612 78689 | 85 31 | 3.02281 86887 | 71 |
| 0.90830 41205 | 3.08742 47870 | 0.21678 03419 | 85 13 | 2.98024 37776 | 70 |
| 0.89935 26403 | 3.05836 91177 | 0.22739 68349 | 84 55 | 2.93766 88665 | 69 |
| 0.89005 01452 | 3.02822 03368 | 0.23797 39802 | 84 37 | 2.89509 39554 | 68 |
| 0.88040 78152 | 2.99702 15345 | 0.24850 81357 | 84 18 | 2.85251 90443 | 67 |
| 0.87043 71170 | 2.96481 70925 | 0.25899 53603 | 83 58 | 2.80994 41332 | 66 |
| 0.86014 97763 | 2.93165 25995 | 0.26943 13876 | 83 38 | 2.76736 92221 | 65 |
| 0.84955 77491 | 2.89757 47641 | 0.27981 15977 | 83 17 | 2.72479 43110 | 64 |
| 0.83867 31932 | 2.86263 13272 | 0.29013 09871 | 82 55 | 2.68221 93999 | 63 |
| 0.82750 84383 | 2.82687 09732 | 0.30038 41353 | 82 33 | 2.63964 44888 | 62 |
| 0.81607 59576 | 2.79034 32412 | 0.31056 51708 | 82 10 | 2.59706 95776 | 61 |
| 0.80438 83372 | 2.75309 84351 | 0.32066 77330 | 81 46 | 2.55449 46665 | 60 |
| 0.79245 82474 | 2.71518 75345 | 0.33068 49323 | 81 21 | 2.51191 97554 | 59 |
| 0.78029 84129 | 2.67666 21047 | 0.34060 93073 | 80 55 | 2.46934 48443 | 58 |
| 0.76792 15834 | 2.63757 42081 | 0.35043 27789 | 80 28 | 2.42676 99332 | 57 |
| 0.75534 05043 | 2.59797 63158 | 0.36014 66018 | 80 0 | 2.38419 50221 | 56 |
| 0.74256 78883 | 2.55792 12198 | 0.36974 13124 | 79 31 | 2.34162 01110 | 55 |
| 0.72961 63864 | 2.51746 19471 | 0.37920 66740 | 79 2 | 2.29904 51999 | 54 |
| 0.71649 85603 | 2.47665 16742 | 0.38853 16185 | 78 30 | 2.25647 02888 | 53 |
| 0.70322 68545 | 2.43554 36438 | 0.39770 41848 | 77 58 | 2.21389 53777 | 52 |
| 0.68981 35699 | 2.39419 10827 | 0.40671 14546 | 77 24 | 2.17132 04666 | 51 |
| 0.67627 08370 | 2.35264 71220 | 0.41553 94843 | 76 50 | 2.12874 55554 | 50 |
| 0.66261 05910 | 2.31096 47190 | 0.42417 32345 | 76 13 | 2.08617 06443 | 49 |
| 0.64884 45467 | 2.26919 65819 | 0.43259 64967 | 75 35 | 2.04359 57332 | 48 |
| 0.63498 41750 | 2.22739 50955 | 0.44079 18172 | 74 56 | 2.00102 08221 | 47 |
| 0.62104 06800 | 2.18561 22515 | 0.44874 04204 | 74 16 | 1.95844 59110 | 46 |
| 0.60702 49768 | 2.14389 95792 | 0.45642 21286 | 73 33 | 1.91587 09999 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

$K = 4.0527581695$, $K' = 1.5727124350$, $E = 1.0086479569$, $E' = 1.5688837196$,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.04503 06463 | 2 35 | 0.03379 31823 | 1.00076 14948 | 0.01189 42847 |
| 2 | 0.09006 12927 | 5 9 | 0.06740 53633 | 1.00304 53671 | 0.02379 47903 |
| 3 | 0.13509 19390 | 7 43 | 0.10065 84494 | 1.00684 97794 | 0.03570 77106 |
| 4 | 0.18012 25853 | 10 16 | 0.13338 00630 | 1.01217 16668 | 0.04763 91855 |
| 5 | 0.22515 32316 | 12 48 | 0.16540 61602 | 1.01900 67332 | 0.05959 52742 |
| 6 | 0.27018 38780 | 15 18 | 0.19658 33739 | 1.02734 94459 | 0.07158 19286 |
| 7 | 0.31521 45243 | 17 46 | 0.22677 10168 | 1.03719 30291 | 0.08360 49670 |
| 8 | 0.36024 51706 | 20 13 | 0.25584 26948 | 1.04852 94558 | 0.09567 00478 |
| 9 | 0.40527 58170 | 22 37 | 0.28368 75021 | 1.06134 94387 | 0.10778 26441 |
| 10 | 0.45030 64633 | 24 58 | 0.31021 07894 | 1.07564 24197 | 0.11994 80182 |
| 11 | 0.49533 71096 | 27 18 | 0.33533 45137 | 1.09139 65585 | 0.13217 11972 |
| 12 | 0.54036 77559 | 29 34 | 0.35899 71966 | 1.10859 87206 | 0.14445 69485 |
| 13 | 0.58539 84023 | 31 47 | 0.38115 35291 | 1.12723 44637 | 0.15680 97563 |
| 14 | 0.63042 90486 | 33 57 | 0.40177 36714 | 1.14728 80243 | 0.16923 37988 |
| 15 | 0.67545 96949 | 36 4 | 0.42084 23033 | 1.16874 23039 | 0.18173 29260 |
| 16 | 0.72049 03413 | 38 8 | 0 43835 74800 | 1.19157 88539 | 0.19431 06384 |
| 17 | 0.76552 09876 | 40 8 | 0.45432 93515 | 1.21577 78616 | 0.20697 00661 |
| 18 | 0.81055 16339 | 42 5 | 0.46877 87966 | 1.24131 81358 | 0.21971 39498 |
| 19 | 0.85558 22802 | 43 58 | 0.48173 60209 | 1.26817 70925 | 0.23254 46217 |
| 20 | 0.90061 29266 | 45 53 | 0.49323 91602 | 1.29633 07415 | 0.24546 39877 |
| 21 | 0.94564 35729 | 47 35 | 0.50333 29227 | 1.32575 36734 | 0.25847 35115 |
| 22 | 0.99067 42192 | 49 18 | 0.51206 72988 | 1.35641 90478 | 0.27157 41984 |
| 23 | 1.03570 48656 | 50 57 | 0.51949 63591 | 1.38829 85826 | 0.28476 65811 |
| 24 | 1.08073 55119 | 52 33 | 0.52567 71528 | 1.42136 25446 | 0.29805 07071 |
| 25 | 1.12576 61582 | 54 6 | 0.53066 87177 | 1.45557 97413 | 0.31142 61261 |
| 26 | 1.17079 68045 | 55 36 | 0.53453 12033 | 1.49091 75157 | 0.32489 18800 |
| 27 | 1.21582 74509 | 57 2 | 0.53732 51072 | 1.52734 17416 | 0.33844 64932 |
| 28 | 1.26085 80972 | 58 25 | 0.53911 06227 | 1.56481 68225 | 0.35208 79650 |
| 29 | 1.30588 87435 | 59 45 | 0.53994 70893 | 1.60330 56919 | 0.36581 37630 |
| 30 | 1.35091 93898 | 61 2 | 0.53989 25408 | 1.64276 98172 | 0.37962 08180 |
| 31 | 1.39595 00362 | 62 16 | 0.53900 33421 | 1.68316 92055 | 0.39350 55205 |
| 32 | 1.44098 06825 | 63 28 | 0.53733 39051 | 1.72446 24133 | 0.40746 37182 |
| 33 | 1.48601 13288 | 64 36 | 0.53493 64751 | 1.76660 65590 | 0.42149 07161 |
| 34 | 1.53104 19752 | 65 42 | 0.53186 09786 | 1.80955 73388 | 0.43558 12766 |
| 35 | 1.57607 26215 | 66 45 | 0.52815 49246 | 1.85326 90463 | 0.44972 96226 |
| 36 | 1.62110 32678 | 67 46 | 0.52386 33506 | 1.89769 45959 | 0.46392 94409 |
| 37 | 1.66613 39141 | 68 44 | 0.51902 88062 | 1.94278 55494 | 0.47817 38881 |
| 38 | 1.71116 45605 | 69 40 | 0.51369 13678 | 1.98849 21476 | 0.49245 55978 |
| 39 | 1.75619 52068 | 70 33 | 0.50788 86793 | 2.03476 33449 | 0.50676 66888 |
| 40 | 1.80122 58531 | 71 25 | 0.50165 60117 | 2.08154 68491 | 0.52109 87757 |
| 41 | 1.84625 64995 | 72 14 | 0.49502 63387 | 2.12878 91642 | 0.53544 29804 |
| 42 | 1.89128 71458 | 73 2 | 0.48803 04242 | 2.17643 56384 | 0.54978 99455 |
| 43 | 1.93631 77921 | 73 47 | 0.48069 69176 | 2.22443 05163 | 0.56412 98491 |
| 44 | 1.98134 84385 | 74 31 | 0.47305 24550 | 2.27271 69945 | 0.57845 24208 |
| 45 | 2.02637 90848 | 75 12 | 0.46512 17631 | 2.32123 72832 | 0.59274 69597 |
| 90−r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 86°$

$q = 0.295488385558687$, $\Theta 0 = 0.4242361430$, $HK = 1.6043008048$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90 r |
|---|---|---|---|---|---|
| 1.00000 00000 | 3.78623 65254 | 0.00000 00000 | 90° 0′ | 4.05275 81695 | 90 |
| 0.99974 76964 | 3.78529 99318 | 0.01098 79345 | 89 49 | 4.00772 75232 | 89 |
| 0.99899 11477 | 3.78249 16163 | 0.02197 49829 | 89 38 | 3.96269 68769 | 88 |
| 0.99773 14382 | 3.77781 59714 | 0.03296 02520 | 89 28 | 3.91766 62306 | 87 |
| 0.99597 03726 | 3.77128 03065 | 0.04394 28343 | 89 17 | 3.87263 55842 | 86 |
| 0.99371 04703 | 3.76289 48312 | 0.05492 18007 | 89 6 | 3.82760 49379 | 85 |
| 0.99095 49588 | 3.75267 26317 | 0.06589 61931 | 88 54 | 3.78257 42916 | 84 |
| 0.98770 77652 | 3.74062 96405 | 0.07686 50165 | 88 43 | 3.73754 36452 | 83 |
| 0.98397 35058 | 3.72678 46000 | 0.08782 72314 | 88 32 | 3.69251 29989 | 82 |
| 0.97975 74732 | 3.71115 90191 | 0.09878 17452 | 88 20 | 3.64748 23526 | 81 |
| 0.97506 56227 | 3.69377 71248 | 0.10972 74034 | 88 8 | 3.60245 17063 | 80 |
| 0.96990 45558 | 3.67466 58061 | 0.12066 29807 | 87 56 | 3.55742 10599 | 79 |
| 0.96428 15032 | 3.65385 45535 | 0.13158 71709 | 87 44 | 3.51239 04136 | 78 |
| 0.95820 43054 | 3.63137 53926 | 0.14249 85767 | 87 32 | 3.46735 97673 | 77 |
| 0.95168 13914 | 3.60726 28114 | 0.15339 56986 | 87 19 | 3.42232 91209 | 76 |
| 0.94472 17573 | 3.58155 36840 | 0.16427 69227 | 87 5 | 3.37729 84746 | 75 |
| 0.93733 49419 | 3.55428 71880 | 0.17514 05085 | 86 52 | 3.33226 78283 | 74 |
| 0.92953 10017 | 3.52550 47184 | 0.18598 45746 | 86 38 | 3.28723 71820 | 73 |
| 0.92132 04850 | 3.49524 97967 | 0.19680 70842 | 86 24 | 3.24220 65356 | 72 |
| 0.91271 44039 | 3.46356 79762 | 0.20760 58292 | 86 9 | 3.19717 58893 | 71 |
| 0.90372 42062 | 3.43050 67437 | 0.21837 84126 | 85 54 | 3.15214 52430 | 70 |
| 0.89436 17453 | 3.39611 54178 | 0.22912 22300 | 85 38 | 3.10711 45967 | 69 |
| 0.88463 92502 | 3.36044 50445 | 0.23983 44495 | 85 22 | 3.06208 39503 | 68 |
| 0.87456 92937 | 3.32354 82896 | 0.25051 19896 | 85 5 | 3.01705 33040 | 67 |
| 0.86416 47610 | 3.28547 93300 | 0.26115 14957 | 84 48 | 2.97202 26577 | 66 |
| 0.85343 88167 | 3.24629 37417 | 0.27174 93142 | 84 30 | 2.92699 20113 | 65 |
| 0.84240 48716 | 3.20604 83874 | 0.28230 14649 | 84 11 | 2.88196 13650 | 64 |
| 0.83107 65499 | 3.16480 13024 | 0.29280 36106 | 83 52 | 2.83693 07187 | 63 |
| 0.81946 76545 | 3.12261 15798 | 0.30325 10250 | 83 32 | 2.79190 00724 | 62 |
| 0.80759 21336 | 3.07953 92551 | 0.31363 85568 | 83 11 | 2.74686 94260 | 61 |
| 0.79546 40466 | 3.03564 51912 | 0.32396 05923 | 82 49 | 2.70183 87797 | 60 |
| 0.78309 75297 | 2.99099 09630 | 0.33421 10135 | 82 26 | 2.65680 81334 | 59 |
| 0.77050 67624 | 2.94563 87432 | 0.34438 31544 | 82 3 | 2.61177 74870 | 58 |
| 0.75770 59335 | 2.89965 11884 | 0.35446 97527 | 81 39 | 2.56674 68407 | 57 |
| 0.74470 92077 | 2.85309 13269 | 0.36446 28984 | 81 13 | 2.52171 61944 | 56 |
| 0.73153 06927 | 2.80602 24483 | 0.37435 39786 | 80 47 | 2.47668 55480 | 55 |
| 0.71818 44065 | 2.75850 79940 | 0.38413 36176 | 80 19 | 2.43165 49017 | 54 |
| 0.70468 42455 | 2.71061 14508 | 0.39379 16142 | 79 50 | 2.38662 42554 | 53 |
| 0.69104 39537 | 2.66239 62465 | 0.40331 68729 | 79 20 | 2.34159 36091 | 52 |
| 0.67727 70914 | 2.61392 56481 | 0.41269 73321 | 78 49 | 2.29656 29627 | 51 |
| 0.66339 70061 | 2.56526 26633 | 0.42191 98869 | 78 17 | 2.25153 23164 | 50 |
| 0.64941 68038 | 2.51646 99446 | 0.43097 03076 | 77 43 | 2.20650 16701 | 49 |
| 0.63534 93209 | 2.46760 96971 | 0.43983 31542 | 77 8 | 2.16147 10238 | 48 |
| 0.62120 70978 | 2.41874 35896 | 0.44849 16855 | 76 31 | 2.11644 03774 | 47 |
| 0.60700 23531 | 2.36993 26700 | 0.45692 77651 | 75 52 | 2.07140 97311 | 46 |
| 0.59274 69597 | 2.32123 72832 | 0.46512 17631 | 75 12 | 2.02637 90848 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 4.3386539760, K′ = 1.5718736105, E = 1.0052585872, E′ = 1.5697201504,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.04820 72664 | 2 46 | 0.03700 05198 | 1.00089 26934 | 0.01102 97158 |
| 2 | 0.09641 45328 | 5 31 | 0.07377 86246 | 1.00357 01695 | 0.02206 73089 |
| 3 | 0.14462 17992 | 8 15 | 0.11011 59944 | 1.00803 06141 | 0.03312 06260 |
| 4 | 0.19282 90656 | 10 59 | 0.14580 23384 | 1.01427 09982 | 0.04419 74541 |
| 5 | 0.24103 63320 | 13 41 | 0.18063 90239 | 1.02228 70707 | 0.05530 54893 |
| 6 | 0.28924 35984 | 16 21 | 0.21444 22668 | 1.03207 33471 | 0.06645 23081 |
| 7 | 0.33745 08648 | 18 59 | 0.24704 57854 | 1.04362 30963 | 0.07764 53371 |
| 8 | 0.38565 81312 | 21 34 | 0.27830 28485 | 1.05692 83239 | 0.08889 18239 |
| 9 | 0.43386 53976 | 24 7 | 0.30808 76822 | 1.07197 97531 | 0.10019 88085 |
| 10 | 0.48207 26640 | 26 37 | 0.33629 62369 | 1.08876 68032 | 0.11157 30946 |
| 11 | 0.53027 99304 | 29 3 | 0.36284 63422 | 1.10727 75652 | 0.12302 12218 |
| 12 | 0.57848 71968 | 31 27 | 0.38767 73064 | 1.12749 87762 | 0.13454 94383 |
| 13 | 0.62669 44632 | 33 46 | 0.41074 90335 | 1.14941 57909 | 0.14616 36738 |
| 14 | 0.67490 17296 | 36 2 | 0.43204 07437 | 1.17301 25520 | 0.15786 95139 |
| 15 | 0.72310 89960 | 38 14 | 0.45154 93887 | 1.19827 15591 | 0.16967 21746 |
| 16 | 0.77131 62624 | 40 23 | 0.46928 78534 | 1.22517 38362 | 0.18157 64776 |
| 17 | 0.81952 35288 | 42 27 | 0.48528 30289 | 1.25369 88987 | 0.19358 68272 |
| 18 | 0.86773 07952 | 44 28 | 0.49957 38349 | 1.28382 47193 | 0.20570 71870 |
| 19 | 0.91593 80616 | 46 24 | 0.51220 92565 | 1.31552 76945 | 0.21794 10587 |
| 20 | 0 96414 53280 | 48 16 | 0.52324 64512 | 1.34878 26100 | 0.23029 14612 |
| 21 | 1.01235 25944 | 50 5 | 0.53274 89656 | 1.38356 26077 | 0.24276 09111 |
| 22 | 1.06055 98608 | 51 50 | 0.54078 50933 | 1.41983 91529 | 0.25535 14044 |
| 23 | 1.10876 71272 | 53 30 | 0.54742 63924 | 1.45758 20021 | 0.26806 43994 |
| 24 | 1.15697 43936 | 55 7 | 0.55274 63730 | 1.49675 91734 | 0.28090 08008 |
| 25 | 1.20518 16600 | 56 40 | 0.55681 93566 | 1.53733 69175 | 0.29386 09452 |
| 26 | 1.25338 89264 | 58 10 | 0.55971 95044 | 1.57927 96919 | 0.30694 45879 |
| 27 | 1.30159 61928 | 59 36 | 0.56152 00057 | 1.62255 01370 | 0.32015 08913 |
| 28 | 1.34980 34592 | 60 58 | 0.56229 24153 | 1.66710 90551 | 0.33347 84147 |
| 29 | 1.39801 07256 | 62 17 | 0.56210 61265 | 1.71291 53925 | 0.34692 51057 |
| 30 | 1.44621 79920 | 63 33 | 0.56102 79658 | 1.75992 62260 | 0.36048 82928 |
| 31 | 1.49442 52584 | 64 46 | 0.55912 18929 | 1.80809 67519 | 0.37416 46804 |
| 32 | 1.54263 25248 | 65 55 | 0.55644 87947 | 1.85738 02804 | 0.38795 03444 |
| 33 | 1.59083 97912 | 67 2 | 0.55306 63561 | 1.90772 82336 | 0.40184 07305 |
| 34 | 1.63904 70676 | 68 6 | 0.54902 89975 | 1.95909 01488 | 0.41583 06538 |
| 35 | 1.68725 43240 | 69 7 | 0.54438 78661 | 2.01141 36867 | 0.42991 42995 |
| 36 | 1.73546 15904 | 70 5 | 0.53919 08711 | 2.06464 46451 | 0.44408 52267 |
| 37 | 1.78366 88568 | 71 1 | 0.53348 27539 | 2.11872 69773 | 0.45833 63730 |
| 38 | 1.83187 61232 | 71 54 | 0.52730 51847 | 2.17360 28173 | 0.47266 00609 |
| 39 | 1.88008 33896 | 72 45 | 0.52069 68791 | 2.22921 25107 | 0.48704 80065 |
| 40 | 1.92829 06560 | 73 34 | 0.51369 37297 | 2.28549 46508 | 0.50149 13298 |
| 41 | 1.97649 79224 | 74 20 | 0.50632 89466 | 2.34238 61220 | 0.51598 05665 |
| 42 | 2.02470 51888 | 75 5 | 0.49863 32034 | 2.39982 21493 | 0.53050 56822 |
| 43 | 2.07291 24552 | 75 47 | 0.49063 47860 | 2.45773 63538 | 0.54505 60878 |
| 44 | 2.12111 97216 | 76 58 | 0.48235 97411 | 2.51606 08149 | 0.55962 06569 |
| 45 | 2.16932 69880 | 77 7 | 0.47383 20219 | 2.57472 61393 | 0.57418 77451 |
| 90−r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.320400337134867$, $\Theta 0 = 0.3802048484$, $HK = 1.6608093153$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90–r |
|---|---|---|---|---|---|
| 1.00000 00000 | 4.37119 23556 | 0.00000 00000 | 90° 0′ | 4.33865 39760 | 90 |
| 0.99973 08085 | 4.37002 95871 | 0.01103 73956 | 89 51 | 4.29044 67096 | 89 |
| 0.99892 36540 | 4.36654 32014 | 0.02207 41777 | 89 43 | 4.24223 94432 | 88 |
| 0.99757 97949 | 4.36073 89539 | 0.03310 97273 | 89 34 | 4.19403 21768 | 87 |
| 0.99570 13248 | 4.35262 64203 | 0.04414 34137 | 89 25 | 4.14582 49104 | 86 |
| 0.99329 11666 | 4.34221 89731 | 0.05517 45893 | 89 16 | 4.09761 76440 | 85 |
| 0.99035 30638 | 4.32953 37471 | 0.06620 25830 | 89 7 | 4.04941 03776 | 84 |
| 0.98689 15704 | 4.31459 15972 | 0.07722 66944 | 88 58 | 4.00120 31112 | 83 |
| 0.98291 20378 | 4.29741 70454 | 0.08824 61873 | 88 49 | 3.95299 58448 | 82 |
| 0.97842 05999 | 4.27803 82196 | 0.09926 02826 | 88 39 | 3.90478 85784 | 81 |
| 0.97342 41557 | 4.25648 67836 | 0.11026 81515 | 88 30 | 3.85658 13120 | 80 |
| 0.96793 03503 | 4.23279 78580 | 0.12126 89076 | 88 20 | 3.80837 40456 | 79 |
| 0.96194 75529 | 4.20700 99336 | 0.13226 15989 | 88 10 | 3.76016 67792 | 78 |
| 0.95548 48341 | 4.17916 47765 | 0.14324 51989 | 88 0 | 3.71195 95128 | 77 |
| 0.94855 19406 | 4.14930 73254 | 0.15421 85972 | 87 49 | 3.66375 22464 | 76 |
| 0.94115 92676 | 4.11748 55826 | 0.16518 05896 | 87 38 | 3.61554 49800 | 75 |
| 0.93331 78308 | 4.08375 04971 | 0.17612 98666 | 87 27 | 3.56733 77136 | 74 |
| 0.92503 92359 | 4.04815 58427 | 0.18706 50017 | 87 16 | 3.51913 04472 | 73 |
| 0.91633 56463 | 4.01075 80891 | 0.19798 44386 | 87 4 | 3.47092 31808 | 72 |
| 0.90721 97509 | 3.97161 62682 | 0.20888 64763 | 86 51 | 3.42271 59144 | 71 |
| 0.89770 47288 | 3.93079 18356 | 0.21976 92546 | 86 38 | 3.37450 86480 | 70 |
| 0.88780 42140 | 3.88834 85274 | 0.23063 07363 | 86 25 | 3.32630 13816 | 69 |
| 0.87753 22590 | 3.84435 22135 | 0.24146 86896 | 86 11 | 3.27809 41152 | 68 |
| 0.86690 32971 | 3.79887 07472 | 0.25228 06673 | 85 57 | 3.22988 68488 | 67 |
| 0.85593 21039 | 3.75197 38123 | 0.26306 39853 | 85 42 | 3.18167 95824 | 66 |
| 0.84463 37589 | 3.70373 27678 | 0.27381 56982 | 85 27 | 3.13347 23160 | 65 |
| 0.83302 36055 | 3.65422 04910 | 0.28453 25731 | 85 11 | 3.08526 50496 | 64 |
| 0.82111 72113 | 3.60351 12193 | 0.29521 10610 | 84 54 | 3.03705 77832 | 63 |
| 0.80893 03281 | 3.55168 03915 | 0.30584 72655 | 84 37 | 2.98885 05168 | 62 |
| 0.79647 88516 | 3.49880 44891 | 0.31643 69081 | 84 19 | 2.94064 32504 | 61 |
| 0.78377 87810 | 3.44496 08773 | 0.32697 52911 | 84 0 | 2.89243 59840 | 60 |
| 0.77084 61787 | 3.39022 76481 | 0.33745 72566 | 83 40 | 2.84422 87176 | 59 |
| 0.75769 71307 | 3.33468 34641 | 0.34787 71421 | 83 19 | 2.79602 14512 | 58 |
| 0.74434 77069 | 3.27840 74042 | 0.35822 87319 | 82 57 | 2.74781 41848 | 57 |
| 0.73081 39218 | 3.22147 88118 | 0.36850 52042 | 82 35 | 2.69960 69184 | 56 |
| 0.71711 16962 | 3.16397 71463 | 0.37869 90740 | 82 11 | 2.65139 96520 | 55 |
| 0.70325 68193 | 3.10598 18371 | 0.38880 21304 | 81 47 | 2.60319 23856 | 54 |
| 0.68926 49116 | 3.04757 21420 | 0.39880 53693 | 81 21 | 2.55498 51192 | 53 |
| 0.67515 13887 | 2.98882 70090 | 0.40869 89202 | 80 54 | 2.50677 78528 | 52 |
| 0.66093 14267 | 2.92982 49435 | 0.41847 19672 | 80 26 | 2.45857 05864 | 51 |
| 0.64661 99275 | 2.87064 38790 | 0.42811 26638 | 79 56 | 2.41036 33200 | 50 |
| 0.63223 14865 | 2.81136 10542 | 0.43760 80415 | 79 25 | 2.36215 60536 | 49 |
| 0.61778 03606 | 2.75205 28945 | 0.44694 39111 | 78 53 | 2.31394 87872 | 48 |
| 0.60328 04384 | 2.69279 48995 | 0.45610 47583 | 78 19 | 2.26574 15208 | 47 |
| 0.58874 52110 | 2.63366 15364 | 0.46507 36311 | 77 44 | 2.21753 42544 | 46 |
| 0.57418 77451 | 2.57472 61393 | 0.47383 20219 | 77 7 | 2.16932 69880 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 4.7427172653, K′ = 1.5712749524, E = 1.0025840855, E′ = 1.5703179199,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.05269 68585 | 3 1 | 0.04150 83698 | 1.00109 49202 | 0.00984 61866 |
| 2 | 0.10539 37170 | 6 2 | 0.08272 60369 | 1.00437 91719 | 0.01970 23988 |
| 3 | 0.15809 05755 | 9 1 | 0.12336 86879 | 1.00985 12249 | 0.02957 86287 |
| 4 | 0.21078 74340 | 11 59 | 0.16316 44916 | 1.01750 85180 | 0.03948 48012 |
| 5 | 0.26348 42925 | 14 56 | 0.20185 96235 | 1.02734 74434 | 0.04943 07415 |
| 6 | 0.31618 11510 | 17 49 | 0.23922 29917 | 1.03936 33238 | 0.05942 61408 |
| 7 | 0.36887 80095 | 20 40 | 0.27504 99964 | 1.05355 03843 | 0.06948 05245 |
| 8 | 0.42157 48680 | 23 28 | 0.30916 52198 | 1.06990 17180 | 0.07960 32187 |
| 9 | 0.47427 17265 | 26 13 | 0.34142 40166 | 1.08840 92458 | 0.08980 33181 |
| 10 | 0.52696 85850 | 28 53 | 0.37171 30376 | 1.10906 36709 | 0.10008 96542 |
| 11 | 0.57966 54435 | 31 30 | 0.39994 97772 | 1.13185 44282 | 0.11047 07636 |
| 12 | 0.63236 23020 | 34 2 | 0.42608 12751 | 1.15676 96284 | 0.12095 48573 |
| 13 | 0.68505 91605 | 36 30 | 0.45008 21300 | 1.18379 59985 | 0.13154 97896 |
| 14 | 0.73775 60190 | 38 53 | 0.47195 19964 | 1.21291 88175 | 0.14226 30292 |
| 15 | 0.79045 28775 | 41 12 | 0.49171 27333 | 1.24412 18489 | 0.15310 16293 |
| 16 | 0.84314 97360 | 43 26 | 0.50940 53625 | 1.27738 72698 | 0.16407 21997 |
| 17 | 0.89584 65946 | 45 35 | 0.52508 69758 | 1.31269 55975 | 0.17518 08788 |
| 18 | 0.94854 34531 | 47 40 | 0.53882 77072 | 1.35002 56142 | 0.18643 33074 |
| 19 | 1.00124 03116 | 49 40 | 0.55070 78595 | 1.38935 42896 | 0.19783 46027 |
| 20 | 1.05393 71701 | 51 34 | 0.56081 52531 | 1.43065 67027 | 0.20938 93338 |
| 21 | 1.10663 40286 | 53 25 | 0.56924 28378 | 1.47390 59633 | 0.22110 14976 |
| 22 | 1.15933 08871 | 55 11 | 0.57608 65921 | 1.51907 31337 | 0.23297 44971 |
| 23 | 1.21202 77456 | 56 52 | 0.58144 37172 | 1.56612 71505 | 0.24501 11193 |
| 24 | 1.26472 46041 | 58 29 | 0.58541 11188 | 1.61503 47485 | 0.25721 35159 |
| 25 | 1.31742 14626 | 60 2 | 0.58808 41618 | 1.66576 03865 | 0.26958 31846 |
| 26 | 1.37011 83211 | 61 31 | 0.58955 56773 | 1.71826 61750 | 0.28212 09517 |
| 27 | 1.42281 51796 | 62 55 | 0.58991 51945 | 1.77251 18082 | 0.29482 69565 |
| 28 | 1.47551 20381 | 64 16 | 0.58924 83721 | 1.82845 44989 | 0.30770 06377 |
| 29 | 1.52820 88966 | 65 33 | 0.58763 66017 | 1.88604 89185 | 0.32074 07202 |
| 30 | 1.58090 57551 | 66 46 | 0.58515 67551 | 1.94524 71416 | 0.33394 52050 |
| 31 | 1.63360 26136 | 67 56 | 0.58188 10541 | 2.00599 85969 | 0.34731 13599 |
| 32 | 1.68629 94721 | 69 3 | 0.57787 70364 | 2.06825 00238 | 0.36083 57125 |
| 33 | 1.73899 63306 | 70 6 | 0.57320 76019 | 2.13194 54360 | 0.37451 40449 |
| 34 | 1.79169 31891 | 71 7 | 0.56793 11188 | 2.19702 60925 | 0.38834 13902 |
| 35 | 1.84439 00476 | 72 4 | 0.56210 15757 | 2.26343 04764 | 0.40231 20314 |
| 36 | 1.89708 69061 | 72 59 | 0.55576 87678 | 2.33109 42822 | 0.41641 95021 |
| 37 | 1.94978 37646 | 73 51 | 0.54897 85058 | 2.39995 04116 | 0.43065 65890 |
| 38 | 2.00248 06231 | 74 41 | 0.54177 28388 | 2.46992 89791 | 0.44501 53371 |
| 39 | 2.05517 74816 | 75 28 | 0.53419 02851 | 2.54095 73266 | 0.45948 70563 |
| 40 | 2.10787 43401 | 76 12 | 0.52626 60647 | 2.61296 00482 | 0.47406 23311 |
| 41 | 2.16057 11986 | 76 55 | 0.51803 23296 | 2.68585 90255 | 0.48873 10316 |
| 42 | 2.21326 80571 | 77 35 | 0.50951 83887 | 2.75957 34731 | 0.50348 23272 |
| 43 | 2.26596 49156 | 78 14 | 0.50075 09241 | 2.83401 99954 | 0.51830 47025 |
| 44 | 2.31866 17741 | 78 50 | 0.49175 41985 | 2.90911 26530 | 0.53318 59750 |
| 45 | 2.37135 86326 | 79 25 | 0.48255 02516 | 2.98476 30422 | 0.54811 33155 |
| 90–r | Fψ | ψ | G(r) | C(r) | B(r) |

$q = 0.353165648296037$, $\Theta 0 = 0.3246110213$, $HK = 1.7370861537$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90-r |
|---|---|---|---|---|---|
| 1.00000 00000 | 5.35291 58734 | 0.00000 00000 | 90° 0′ | 4.74271 72653 | 90 |
| 0.99970 65254 | 5.35135 39870 | 0.01107 55804 | 89 54 | 4.69002 04068 | 89 |
| 0.99882 66090 | 5.34667 11120 | 0.02215 08037 | 89 47 | 4.63732 35483 | 88 |
| 0.99736 17711 | 5.33887 55928 | 0.03322 53090 | 89 41 | 4.58462 66898 | 87 |
| 0.99531 45401 | 5.32798 13106 | 0.04429 87274 | 89 35 | 4.53192 98313 | 86 |
| 0.99268 84456 | 5.31400 76445 | 0.05537 06778 | 89 28 | 4.47923 29728 | 85 |
| 0.98948 80069 | 5.29697 94165 | 0.06644 07630 | 89 21 | 4.42653 61143 | 84 |
| 0.98571 87199 | 5.27692 68222 | 0.07750 85650 | 89 15 | 4.37383 92558 | 83 |
| 0.98138 70401 | 5.25388 53459 | 0.08857 36405 | 89 8 | 4.32114 23973 | 82 |
| 0.97650 03636 | 5.22789 56618 | 0.09963 55161 | 89 1 | 4.26844 55388 | 81 |
| 0.97106 70046 | 5.19900 35203 | 0.11069 36828 | 88 54 | 4.21574 86803 | 80 |
| 0.96509 61704 | 5.16725 96214 | 0.12174 75905 | 88 46 | 4.16305 18218 | 79 |
| 0.95859 79343 | 5.13271 94744 | 0.13279 66420 | 88 39 | 4.11035 49633 | 78 |
| 0.95158 32050 | 5.09544 32457 | 0.14384 01862 | 88 31 | 4.05765 81048 | 77 |
| 0.94406 36948 | 5.05549 55939 | 0.15487 75112 | 88 23 | 4.00496 12463 | 76 |
| 0.93605 18846 | 5.01294 54947 | 0.16590 78361 | 88 15 | 3.95226 43878 | 75 |
| 0.92756 09875 | 4.96786 60538 | 0.17693 03026 | 88 6 | 3.89956 75293 | 74 |
| 0.91860 49094 | 4.92033 43119 | 0.18794 39654 | 87 58 | 3.84687 06707 | 73 |
| 0.90919 82095 | 4.87043 10392 | 0.19894 77822 | 87 48 | 3.79417 38122 | 72 |
| 0.89935 60570 | 4.81824 05226 | 0.20994 06015 | 87 39 | 3.74147 69537 | 71 |
| 0.88909 41880 | 4.76385 03454 | 0.22092 11507 | 87 29 | 3.68878 00952 | 70 |
| 0.87842 88604 | 4.70735 11607 | 0.23188 80216 | 87 18 | 3.63608 32367 | 69 |
| 0.86737 68071 | 4.64883 64589 | 0.24283 96552 | 87 8 | 3.58338 63782 | 68 |
| 0.85595 51894 | 4.58840 23314 | 0.25377 43247 | 86 56 | 3.53068 95197 | 67 |
| 0.84418 15481 | 4.52614 72300 | 0.26469 01166 | 86 45 | 3.47799 26612 | 66 |
| 0.83207 37552 | 4.46217 17234 | 0.27558 49098 | 86 32 | 3.42529 58027 | 65 |
| 0.81964 99644 | 4.39657 82526 | 0.28645 63526 | 86 19 | 3.37259 89442 | 64 |
| 0.80692 85610 | 4.32947 08849 | 0.29730 18370 | 86 6 | 3.31990 20857 | 63 |
| 0.79392 81128 | 4.26095 50677 | 0.30811 84711 | 85 52 | 3.26720 52272 | 62 |
| 0.78066 73195 | 4.19113 73836 | 0.31890 30470 | 85 37 | 3.21450 83687 | 61 |
| 0.76716 49636 | 4.12012 53075 | 0.32965 20072 | 85 21 | 3.16181 15102 | 60 |
| 0.75343 98604 | 4.04802 69653 | 0.34036 14062 | 85 5 | 3.10911 46517 | 59 |
| 0.73951 08099 | 3.97495 08972 | 0.35102 68681 | 84 48 | 3.05641 77932 | 58 |
| 0.72539 65478 | 3.90100 58247 | 0.36164 35409 | 84 29 | 3.00372 09347 | 57 |
| 0.71111 56987 | 3.82630 04227 | 0.37220 60448 | 84 10 | 2.95102 40762 | 56 |
| 0.69668 67291 | 3.75094 30973 | 0.38270 84160 | 83 51 | 2.89832 72177 | 55 |
| 0.68212 79026 | 3.67504 17706 | 0.39314 40446 | 83 30 | 2.84563 03592 | 54 |
| 0.66745 72351 | 3.59870 36716 | 0.40350 56060 | 83 8 | 2.79293 35007 | 53 |
| 0.65269 24519 | 3.52203 51359 | 0.41378 49862 | 82 44 | 2.74023 66422 | 52 |
| 0.63785 09470 | 3.44514 14133 | 0.42397 31992 | 82 20 | 2.68753 97837 | 51 |
| 0.62294 97425 | 3.36812 64840 | 0.43406 02965 | 81 55 | 2.63484 29252 | 50 |
| 0.60800 54504 | 3.29109 28843 | 0.44403 52686 | 81 28 | 2.58214 60667 | 49 |
| 0.59303 42368 | 3.21414 15421 | 0.45388 59368 | 80 59 | 2.52944 92081 | 48 |
| 0.57805 17864 | 3.13737 16225 | 0.46359 88357 | 80 29 | 2.47675 23496 | 47 |
| 0.56307 32704 | 3.06088 03834 | 0.47315 90851 | 79 58 | 2.42405 54911 | 46 |
| 0.54811 33155 | 2.98476 30422 | 0.48255 02516 | 79 25 | 2.37135 86326 | 45 |
| A(r) | D(r) | E(r) | $\phi$ | F$\phi$ | r |

K = 5.4349098296, K′ = 1.5709159581, E = 1.0007515777, E′ = 1.5706767091,

| r | Fφ | φ | E(r) | D(r) | A(r) |
|---|---|---|---|---|---|
| 0 | 0.00000 00000 | 0° 0′ | 0.00000 00000 | 1.00000 00000 | 0.00000 00000 |
| 1 | 0.06038 78870 | 3 27 | 0.04919 51488 | 1.00148 76066 | 0.00797 98676 |
| 2 | 0.12077 57740 | 6 54 | 0.09795 31901 | 1.00595 04088 | 0.01597 27570 |
| 3 | 0.18116 36610 | 10 19 | 0.14584 95983 | 1.01338 83449 | 0.02399 16544 |
| 4 | 0.24155 15480 | 13 42 | 0.19248 42494 | 1.02380 12862 | 0.03204 94760 |
| 5 | 0.30193 94350 | 17 3 | 0.23749 17959 | 1.03718 89963 | 0.04015 90322 |
| 6 | 0.36232 73220 | 20 19 | 0.28055 00559 | 1.05355 10766 | 0.04833 29925 |
| 7 | 0.42271 52090 | 23 32 | 0.32138 60670 | 1.07288 68948 | 0.05658 38508 |
| 8 | 0.48310 30960 | 26 40 | 0.35977 96610 | 1.09519 55002 | 0.06492 38899 |
| 9 | 0.54349 09830 | 29 43 | 0.39556 46136 | 1.12047 55228 | 0.07336 51472 |
| 10 | 0.60387 88700 | 32 40 | 0.42862 75917 | 1.14872 50597 | 0.08191 93794 |
| 11 | 0.66426 67569 | 35 32 | 0.45890 52450 | 1.17994 15472 | 0.09059 80283 |
| 12 | 0.72465 46439 | 38 18 | 0.48637 98590 | 1.21412 16208 | 0.09941 21860 |
| 13 | 0.78504 25309 | 40 58 | 0.51107 40138 | 1.25126 09628 | 0.10837 25614 |
| 14 | 0.84543 04179 | 43 32 | 0.53304 46717 | 1.29135 41391 | 0.11748 94454 |
| 15 | 0.90581 83049 | 45 59 | 0.55237 70723 | 1.33439 44250 | 0.12677 26784 |
| 16 | 0.96620 61919 | 48 20 | 0.56917 87466 | 1.38037 36227 | 0.13623 16162 |
| 17 | 1.02659 40789 | 50 35 | 0.58357 38857 | 1.42928 18693 | 0.14587 50978 |
| 18 | 1.08698 19659 | 52 44 | 0.59569 82320 | 1.48110 74384 | 0.15571 14129 |
| 19 | 1.14736 98529 | 54 47 | 0.60569 45851 | 1.53583 65353 | 0.16574 82707 |
| 20 | 1.20775 77399 | 56 43 | 0.61370 89715 | 1.59345 30865 | 0.17599 27682 |
| 21 | 1.26814 56269 | 58 35 | 0.61988 74725 | 1.65393 85266 | 0.18645 13603 |
| 22 | 1.32853 35139 | 60 20 | 0.62437 36797 | 1.71727 15815 | 0.19712 98307 |
| 23 | 1.38892 14009 | 62 0 | 0.62730 67243 | 1.78342 80514 | 0.20803 32624 |
| 24 | 1.44930 92879 | 63 35 | 0.62881 98144 | 1.85238 05926 | 0.21916 60113 |
| 25 | 1.50969 71749 | 65 5 | 0.62903 92100 | 1.92409 85022 | 0.23053 16788 |
| 26 | 1.57008 50619 | 66 30 | 0.62808 35657 | 1.99854 75042 | 0.24213 30872 |
| 27 | 1.63047 29489 | 67 51 | 0.62606 35735 | 2.07568 95405 | 0.25397 22556 |
| 28 | 1.69086 08359 | 69 7 | 0.62308 18462 | 2.15548 25676 | 0.26605 03772 |
| 29 | 1.75124 87229 | 70 19 | 0.61923 29878 | 2.23788 03597 | 0.27836 77989 |
| 30 | 1.81163 66099 | 71 27 | 0.61460 38040 | 2.32283 23203 | 0.29092 40017 |
| 31 | 1.87202 44969 | 72 31 | 0.60927 36149 | 2.41028 33038 | 0.30371 75832 |
| 32 | 1.93241 23839 | 73 32 | 0.60331 46378 | 2.50017 34479 | 0.31674 62424 |
| 33 | 1.99280 02709 | 74 29 | 0.59679 24144 | 2.59243 80185 | 0.33000 67656 |
| 34 | 2.05318 81579 | 75 23 | 0.58976 62623 | 2.68700 72681 | 0.34349 50157 |
| 35 | 2.11357 60449 | 76 14 | 0.58228 97341 | 2.78380 63098 | 0.35720 59222 |
| 36 | 2.17396 39318 | 77 2 | 0.57441 10737 | 2.88275 50068 | 0.37113 34754 |
| 37 | 2.23435 18188 | 77 48 | 0.56617 36598 | 2.98376 78796 | 0.38527 07211 |
| 38 | 2.29473 97058 | 78 31 | 0.55761 64315 | 3.08675 40315 | 0.39960 97596 |
| 39 | 2.35512 75928 | 79 11 | 0.54877 42910 | 3.19161 70942 | 0.41414 17461 |
| 40 | 2.41551 54798 | 79 49 | 0.53967 84809 | 3.29825 51932 | 0.42885 68946 |
| 41 | 2.47590 33668 | 80 25 | 0.53035 69362 | 3.40656 09346 | 0.44374 44843 |
| 42 | 2.53629 12538 | 80 58 | 0.52083 46089 | 3.51642 14148 | 0.45879 28694 |
| 43 | 2.59667 91408 | 81 30 | 0.51113 37664 | 3.62771 82525 | 0.47398 94906 |
| 44 | 2.65706 70278 | 82 0 | 0.50127 42646 | 3.74032 76441 | 0.48932 08915 |
| 45 | 2.71745 49148 | 82 28 | 0.49127 37968 | 3.85412 04436 | 0.50477 27366 |
| 90 r | Fψ | ψ | G(r) | C(r) | B(r) |

TABLE $\theta = 89°$

$q = 0.403309306338378$, $\Theta 0 = 0.2457332317$, $HK = 1.8599580878$

| B(r) | C(r) | G(r) | $\psi$ | F$\psi$ | 90−r |
|---|---|---|---|---|---|
| 1.00000 00000 | 7.56958 97180 | 0.00000 00000 | 90° 0′ | 5.43490 98296 | 90 |
| 0.99966 43156 | 7.56705 29325 | 0.01110 10463 | 89 56 | 5.37452 19426 | 89 |
| 0.99865 79343 | 7.55944 77064 | 0.02220 19579 | 89 53 | 5.31413 40556 | 88 |
| 0.99698 28696 | 7.54678 94142 | 0.03330 25985 | 89 49 | 5.25374 61686 | 87 |
| 0.99464 24694 | 7.52910 36233 | 0.04440 28272 | 89 45 | 5.19335 82816 | 86 |
| 0.99164 14052 | 7.50642 60102 | 0.05550 24979 | 89 42 | 5.13297 03946 | 85 |
| 0.98798 56557 | 7.47880 22428 | 0.06660 14556 | 89 38 | 5.07258 25077 | 84 |
| 0.98368 24869 | 7.44628 78301 | 0.07769 95354 | 89 34 | 5.01219 46207 | 83 |
| 0.97874 04272 | 7.40894 79407 | 0.08879 65593 | 89 30 | 4.95180 67337 | 82 |
| 0.97316 92390 | 7.36685 71893 | 0.09989 23340 | 89 26 | 4.89141 88467 | 81 |
| 0.96697 98856 | 7.32009 93943 | 0.11098 66481 | 89 22 | 4.83103 09597 | 80 |
| 0.96018 44944 | 7.26876 73054 | 0.12207 92686 | 89 17 | 4.77064 30727 | 79 |
| 0.95279 63165 | 7.21296 23044 | 0.13316 99380 | 89 13 | 4.71025 51857 | 78 |
| 0.94482 96828 | 7.15279 40797 | 0.14425 83704 | 89 8 | 4.64986 72987 | 77 |
| 0.93629 99559 | 7.08838 02759 | 0.15534 42469 | 89 3 | 4.58947 94117 | 76 |
| 0.92722 34802 | 7.01984 61207 | 0.16642 72118 | 88 58 | 4.52909 15247 | 75 |
| 0.91761 75278 | 6.94732 40301 | 0.17750 68667 | 88 53 | 4.46870 36377 | 74 |
| 0.90750 02426 | 6.87095 31948 | 0.18858 27648 | 88 47 | 4.40831 57507 | 73 |
| 0.89689 05812 | 6.79087 91481 | 0.19965 44048 | 88 41 | 4.34792 78637 | 72 |
| 0.88580 82522 | 6.70725 33191 | 0.21072 12232 | 88 35 | 4.28753 99767 | 71 |
| 0.87427 36532 | 6.62023 25717 | 0.22178 25863 | 88 29 | 4.22715 20897 | 70 |
| 0.86230 78063 | 6.52997 87323 | 0.23283 77807 | 88 22 | 4.16676 42027 | 69 |
| 0.84993 22921 | 6.43665 81080 | 0.24388 60035 | 88 15 | 4.10637 63157 | 68 |
| 0.83716 91826 | 6.34044 09975 | 0.25492 63501 | 88 7 | 4.04598 84287 | 67 |
| 0.82404 09732 | 6.24150 11966 | 0.26595 78012 | 87 59 | 3.98560 05417 | 66 |
| 0.81057 05141 | 6.14001 55012 | 0.27697 92084 | 87 51 | 3.92521 26547 | 65 |
| 0.79678 09414 | 6.03616 32083 | 0.28798 92768 | 87 42 | 3.86482 47677 | 64 |
| 0.78269 56083 | 5.93012 56192 | 0.29898 65471 | 87 33 | 3.80443 68807 | 63 |
| 0.76833 80165 | 5.82208 55452 | 0.30996 93739 | 87 23 | 3.74404 89937 | 62 |
| 0.75373 17477 | 5.71222 68183 | 0.32093 59022 | 87 12 | 3.68366 11067 | 61 |
| 0.73890 03962 | 5.60073 38100 | 0.33188 40408 | 87 1 | 3.62327 32197 | 60 |
| 0.72386 75024 | 5.48779 09576 | 0.34281 14317 | 86 50 | 3.56288 53328 | 59 |
| 0.70865 64877 | 5.37358 23026 | 0.35371 54168 | 86 37 | 3.50249 74458 | 58 |
| 0.69329 05904 | 5.25829 10413 | 0.36459 29992 | 86 24 | 3.44210 95588 | 57 |
| 0.67779 28032 | 5.14209 90885 | 0.37544 08012 | 86 10 | 3.38172 16718 | 56 |
| 0.66218 58136 | 5.02518 66588 | 0.38625 50154 | 85 55 | 3.32133 37848 | 55 |
| 0.64649 19448 | 4.90773 18631 | 0.39703 13507 | 85 40 | 3.26094 58978 | 54 |
| 0.63073 30999 | 4.78991 03252 | 0.40776 49715 | 85 23 | 3.20055 80108 | 53 |
| 0.61493 07081 | 4.67189 48167 | 0.41845 04298 | 85 6 | 3.14017 01238 | 52 |
| 0.59910 56732 | 4.55385 49133 | 0.42908 15883 | 84 47 | 3.07978 22368 | 51 |
| 0.58327 83254 | 4.43595 66732 | 0.43965 15347 | 84 27 | 3.01939 43498 | 50 |
| 0.56746 83750 | 4.31836 23371 | 0.45015 24856 | 84 6 | 2.95900 64628 | 49 |
| 0.55169 48696 | 4.20123 00521 | 0.46057 56791 | 83 44 | 2.89861 85758 | 48 |
| 0.53597 61539 | 4.08471 36196 | 0.47091 12546 | 83 20 | 2.83823 06888 | 47 |
| 0.52032 98326 | 3.96896 22668 | 0.48114 81189 | 82 55 | 2.77784 28018 | 46 |
| 0.50477 27366 | 3.85412 04436 | 0.49127 37968 | 82 28 | 2.71745 49148 | 45 |
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